E-contents on Heat Transfer

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CHAPTER 1

BASICS CONCEPTS AND LAWS

Introduction: Thermal form of energy transferred due to temperature difference is called heat. The subject of heat transfer deals with the rate at which heat is transferred and the temperature distribution within the system. Heat is transferred by three modes namely, conduction, convection and radiation. Conduction refers to heat transfer within a stationary medium (solid or liquid) due to temperature difference. Convection refers to heat transfer between a surface and the moving fluid (liquid or gas) maintained at different temperatures and it involves mass movement of fluid that may be natural or forced. Radiation refers to heat transfer by electromagnetic waves (or photons) in absence of any medium between two surfaces at different temperatures. The knowledge of heat transfer is required for the design of various heat exchanging devices used in different fields of engineering. This chapter briefly describes the basic concepts related to heat transfer, the mechanisms of the three modes of heat transfer and its basic laws.

1.1 Temperature: Temperature is a property of matter which two bodies in thermal equilibrium have in common. It measures the level of heat in a body and indicates relative hotness or coldness with respect to surrounding. Temperature is a scalar quantity, i.e., it has only magnitude. Temperature (T) at a point (x, y, z) at time t in rectangular coordinates can be expressed as $T = f(x, y, z, t)$. Here, symbol f is used to indicate function.

There are two absolute temperature scales, namely, Kelvin scale (K) and Rankine scale (^oR). The important relations for conversion among the Kelvin, degree centigrade (^oC), Fahrenheit (^oF) and Rankine scales (^oR) are as below.

$$K = {}^oC + 273.15;\; {}^oF = \frac{9}{5}\,{}^oC + 32;\; {}^oR = {}^oF + 459.67 = \frac{9}{5}K$$

1.2 Temperature gradient: The change in temperature per unit length or the rate of change of temperature with respect to the distance in the direction of heat transfer is called temperature gradient. The temperature gradient for heat transfer in x-direction is given as below.

$$\frac{\partial T}{\partial x} \text{ or } \frac{dT}{dx} \text{ in } K/m\,(or\ {}^oC/m).$$

The partial and ordinary derivatives of a function are identical when the function depends on a single variable only. Here, $T = T(x)$.

1.3 Heat: Heat is one of the forms of energy that transfers due to the existence of temperature difference. Heat is vector quantity, i.e., it has direction as well as magnitude. Heat flows in the direction of decreasing temperature with a negative temperature gradient. The larger the temperature gradient, the higher is the rate of heat transfer.

1.4 Differences between thermodynamics and heat transfer:

Thermodynamics	Heat Transfer
1. Thermodynamics deals with the amount of heat transfer as a system undergoes a process from one equilibrium state to another. It does	1. Heat transfer deals with the estimation of heat transfer rate to or from a system and the temperature distribution within the system.

not provide any information about the time rate at which the process occurs.	Heat transfer is a non-equilibrium phenomenon as it deals with systems that lack thermal equilibrium.
2. Principally it is based on the two laws of nature namely, the first law and the second law of thermodynamics. The first law of thermodynamics gives conservation of energy, whereas second law gives direction of heat flow.	2. Heat transfer utilizes first and second laws of thermodynamics, Fourier's law of heat conduction, Newton's law of cooling, Stefan-Boltzmann's law of thermal radiation, equation of continuity, Newton's laws of motion, empirical relations for fluid properties and equation of state.

1.5 First law of thermodynamics: The first law of thermodynamics (also known as the conservation of energy principle) states that energy cannot be created or destroyed but can be transformed from one form to another or transferred as heat or work. To apply this law, control volume is to be identified. A control volume is a fixed region in space bounded by a control surface through which heat, work and mass can pass. The useful form of the first law of thermodynamics for heat transfer analysis is given as below.

"The rate at which thermal and mechanical energies enter a control volume plus the rate at which energy generates within that volume minus the rate at which thermal and mechanical energies leave the volume must equal the rate at which internal energy changes or stores inside this volume".

1.6 Heat flux: The rate of heat transfer (Q) per unit area normal to the direction of heat transfer is called heat flux (q). Mathematically it can be given as below.

$$q = \frac{Q}{A} (W/m^2)$$

Here, Q is the rate of heat transfer in $watts\,(W)$ and A is the area normal to the direction of heat transfer in m^2.

1.7 Application areas of heat transfer: Heat transfer analysis is required for the design of various heat transfer equipments. It mainly concerns with the determination of heat transfer rate and the size of a thermal system for a specified temperature difference. Some of the important engineering systems in which principles and methods of heat transfer find applications are: heat exchangers, condensers, evaporators, furnaces, heaters, boilers, turbine systems, refrigerators, air-conditioning, solar collectors, sizing of the nuclear fuel elements, radiators, internal combustion (IC) engines, bearings, electric and electronic devices, and insulation of houses and the steam pipes.

1.8 Types of heat transfer

(i) Steady state heat transfer: When temperature (T) or heat flux (q) at any location of the system does not vary with time (though both quantities may vary from one location to another), the heat transfer through a medium under such condition is called steady state heat transfer (or steady heat transfer).

(ii) Transient (unsteady state) heat transfer: When temperature or heat flux at any location of the system varies with time as well as location, the heat transfer through a medium under such conditions is called transient heat transfer or unsteady state heat transfer (unsteady heat transfer).

(iii) 1-D, 2-D, 3-D and periodic heat transfer: Heat transfer may occur in one, two or three directions in a system. In one-dimensional heat transfer, the temperature is a function of only one space coordinate. For simplicity, most heat transfer problems are solved by one-dimensional (1-D) analysis. When temperature is a function of two space coordinates, heat flow is two-dimensional (2-D). In a three-dimensional (3-D) heat transfer, the temperature is a function of three space coordinates. A special kind of unsteady process in which temperature changes with time in a cyclic manner is called periodic or quasi-steady state heat transfer. In such a case, the temperature at a particular point attains the same value at definite intervals of time, e.g., the walls of cylinder of an internal combustion engine.

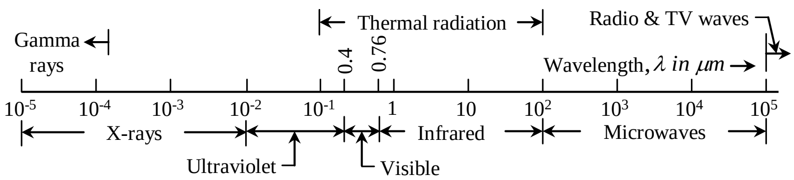
1.9 Modes of heat transfer: There are three modes of heat transfer namely, conduction, convection and radiation.

(i) Conduction: Conduction is the mode of heat transfer in which the medium transporting the heat remains stationary. Heat conduction in solids takes place by the combination of energy transfer by free electrons and molecular vibrations in a lattice (lattice means periodic arrangement of atoms). However, heat conduction through a substance may be viewed as the transfer of energy from high temperature molecules to the adjacent low temperature molecules due to interactions between them. Conduction is the only mode of heat transfer in a solid medium.

(ii) Convection: Heat transfer between a moving fluid (liquid or gas) and a solid surface maintained at different temperatures is called convection. The convection process involves mass movement of fluid that may be natural or forced. Basically it is conduction in a very thin fluid layer adjacent to the heated surface due to random motion of fluid molecules (diffusion) and then its mixing caused by fluid motion (bulk motion).

The convection heat transfer increases with the increase in fluid motion. Heat transfer by convection can be classified as forced or free convection.

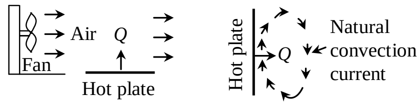
(iii) Radiation Mechanism: In radiation heat transfer, heat is transmitted in the form of electromagnetic waves (or photons) due to changes in the electronic configurations of the atoms or molecules of the matter. It occurs at the speed of light without requiring any intervening medium for its propagation. Heat transfer by radiation occurs most efficiently in vacuum, e.g., energy of sun reaches the earth by radiation. Electromagnetic spectrum in below Figure shows that radiation is emitted over a wide range of wavelengths (λ). The wavelength band of solar radiation is about $0.3\ \mu m$ to $3\ \mu m$.



Electromagnetic spectrum

1.10. Types of convection

(i) Forced convection: When the fluid motion is caused by external means like a blower or fan, the heat transfer is called forced convection. Below Figure illustrates forced convection in which air is forced by a fan to move over a hot plate for its cooling. In forced convection, heat transfer is higher than the natural convection.

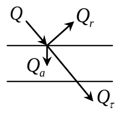


Forced convection **Free convection**

(ii) Free convection: When the fluid motion is caused by buoyancy forces resulting from density differences due to the variation in temperature in the fluid, the heat transfer is termed as natural convection or free convection. Above Figure shows that the stagnant air adjacent to the vertical hot plate heats up and moves up owing to the effect of buoyancy and the nearby cold air moves towards the plate. This motion that results from continuous replacement of heated air in the vicinity of hot plate by the adjacent cold air is called natural convection current and the resulting heat transfer is called as natural convection. Boiling and condensation phenomena (phase change processes) involve the fluid motion, thus these are also considered as convection.

1.11. Thermal radiation: The subject of heat transfer basically deals with thermal radiation which is the energy emitted by a body because of its temperature. All bodies at a temperature of absolute zero (0 K) emit thermal radiation continuously which occurs in the wavelength range of $0.1\ \mu m$ to $100\ \mu m$. All bodies simultaneously emit and absorb radiation. Thermal radiation also depends on optical properties of the body in addition to temperature. For practical purpose, atmospheric air is considered transparent to thermal radiation.

1.12. Absorptivity, reflectivity and transmissivity: Radiation is volumetric phenomenon but it is considered to be a surface phenomenon, since radiation ultimately leaves the surface of the body. Generally, a part of the radiation incident on a surface is absorbed in the body and the rest is reflected and or transmitted. If Q is the rate of total radiant energy incident upon the surface of a body, some part of it will be absorbed (Q_a), some will be reflected (Q_r) and some will be transmitted (Q_τ) through the body (See below figure).



$$Q = Q_a + Q_r + Q_\tau \text{ or } \frac{Q_a}{Q} + \frac{Q_r}{Q} + \frac{Q_\tau}{Q} = 1 \text{ or } \alpha + \rho + \tau = 1$$

Here, α is the asorptivity, ρ is the reflectivity and τ is the transmissivity.

1.13. Fourier's law of heat conduction

Assumptions for Fourier's law: (i) There is steady heat conduction, (ii) Heat flows in one direction only, (iii) Temperature profile is linear thus temperature gradient is constant, (iv) Material is homogeneous (i.e., constant density) and isotropic (i.e., thermal conductivity is same in all directions), (v) The two faces of bounding surfaces are isothermal, and (vi) There is no internal heat generation.

Fourier's law: This law states that the rate of heat conduction is proportional to the area measured normal to the direction of heat flow and to the temperature gradient in that direction. Let Q be the rate of heat conduction in $watts\,(W)$, A be the area normal to the direction of heat flow in m^2 and (dT/dx) be the temperature gradient in K/m. Then according to Fourier's law, we have the following expression.

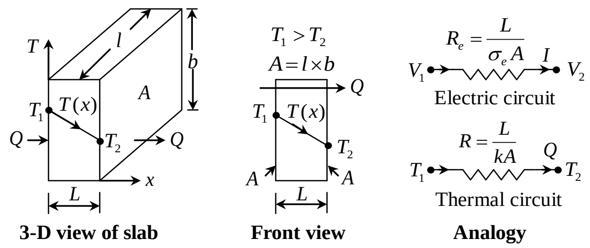
$$Q \propto A\frac{dT}{dx} \text{ or } Q = -kA\frac{dT}{dx}$$

Here, the constant of proportionality k in above equation is called the thermal conductivity of the material. Since heat flows in the direction of decreasing temperature, thus temperature gradient is negative. Therefore, negative sign is introduced in above Equation to make heat transfer a positive quantity.

1.14. Isotropic and anisotropic materials: A material whose properties like thermal conductivity (k), specific heat (c) and density (ρ) remain same in all directions is called isotropic material. In heat transfer analysis usually a material is assumed as isotropic.

A material whose properties depend upon the direction is called anisotropic material, e.g., wood whose thermal conductivity across the grain is different than that parallel to the grain. Other examples of anisotropic materials are laminated composite materials, graphite, fiber-reinforced polymers and asbestos.

1.15. Concept of thermal resistance for a solid metallic slab: Consider steady heat conduction through a plane rectangular wall or slab of thickness L having a constant cross-sectional area A. Let k be the constant thermal conductivity of the wall and $T(x)$ be the temperature distribution. The faces of the wall are maintained at uniform temperatures T_1 and T_2 such that $T_1 > T_2$ as shown in below Figure.



The temperature gradient for the given wall can be expressed as below.

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Thus, the rate of heat conduction through the wall is given by Fourier's law as below.

$$Q = -kA\frac{dT}{dx} = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} = kA\frac{\Delta T}{L}$$

or $$Q = \frac{\Delta T}{L/(kA)} = \frac{\Delta T}{R} = \frac{\text{Temperature difference}}{\text{Conductive thermal resistance}} \quad \text{(i)}$$

The term $L/(kA)$ in above equation is called the conductive thermal resistance $(R_{cond} \; or \; R)$ or simply thermal resistance and it is measured in $K/W \; (or \; {}^oC/W)$.

(i) Ohm's law: The electric current (I) flowing through a conductor of length L in m, cross-sectional area A in m^2 and electrical conductivity σ_e in $\Omega^{-1}m^{-1}$ (Ω stands for ohm) is given by the Ohm's law as below.

$$I = \frac{V_1 - V_2}{L/(\sigma_e A)} = \frac{\Delta V}{R_e} = \frac{\text{Voltage difference}}{\text{Electrical resistance}} \qquad \text{(ii)}$$

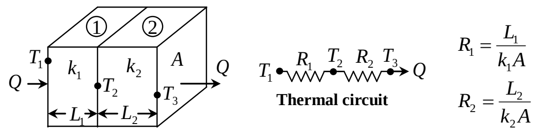
(ii) Analogy between the flow of heat and current: The analogy between the flow of heat through a thermal resistance and the flow of electric current through an electric resistance is shown by thermal circuit and electric circuit, respectively in above figure. Comparison of equations (i) and (ii) for heat flow and current flow suggests that temperature difference is the driving potential for heat flow, analogous to voltage difference being the driving potential for current flow. Then, the conductive thermal resistance to heat flow, $R = L/(kA)$ is analogous to electrical resistance, $R_e = L/(\sigma_e A)$.

(iii) Advantage of the analogy between the flow of heat and current: The advantage of introducing the concept of thermal resistance is that the rule for combining electrical resistances in series and parallel is also applicable to thermal resistances. Thus, complex problems involving both series and parallel thermal resistances can be solved easily, e.g., in a composite wall (that refers to a wall of a several heterogeneous layers), walls of a furnace and boilers.

(iv) Thermal conductance: The reciprocal of the conductive thermal resistance is called as the thermal conductance (K).

Thus $\quad K = \frac{1}{R} = \frac{1}{L/(kA)} = \frac{kA}{L}$

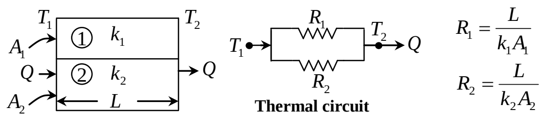
(v) Thermal resistances in series: Considering perfect contact at the interface of the two layers 1 and 2 of a composite wall having thermal resistances R_1 and R_2, respectively.



The total thermal resistance $\left(\sum R\right)$ for conduction through these two thermal resistances in series shown in above figure using rules for combining electrical resistances can be given as follows.

$\sum R = R_1 + R_2$, Here, $R_1 = L_1/(k_1 A)$ and $R_2 = L_2/(k_2 A)$

(vi) Thermal resistances in parallel:



The equivalent thermal resistance (R_{eq}) for conduction through a composite wall containing two thermal resistances in parallel shown in above Figure can be given as below.

$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{eq} = \frac{1}{1/R_1 + 1/R_2}$, Here, $R_1 = L/(k_1 A_1)$ and $R_2 = L/(k_2 A_2)$

1.16. Thermal conductivity

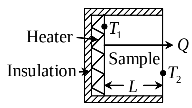
(i) Definition of thermal conductivity and its units: Thermal conductivity is the property of a material that measures its ability to conduct heat. A material of high thermal conductivity is a good heat conductor, whereas a material of low thermal conductivity is a poor heat conductor (or insulator).

The units for thermal conductivity can be obtained from $Q = \frac{\Delta T}{L/(kA)}$ as,

$$k = \frac{Q \times L}{A \times \Delta T} = \frac{W \times m}{m^2 \times K} = \frac{W}{mK}$$

Thermal conductivity is expressed in $W/mK \;\; (or\, W/m\,^oC)$. Thermal conductivity can be defined from above Equation as the rate of heat transfer through a unit thickness of a material per unit area per unit temperature difference.

(ii) Measurement of thermal conductivity for a metallic slab: Let a layer of a material (sample) of thickness L in m and area A in m^2 be heated from one side by an electric resistance heater as shown in below Figure. The outer surface of the heater is perfectly insulated.



(i) Two thermocouples embedded into the surfaces of the sample material measure the temperature values T_1 and T_2.

(ii) The heat generated by the heater (Q) transfers through the material equals the electric power drawn by it can be given as, $Q = (V \times I)\; watts$. Here, V is the voltage in volts and I is the electric current in amperes.

(iii) When steady state is attained the readings for temperatures T_1 and T_2 are recorded and consequently, $\Delta T = (T_1 - T_2)$ is obtained.

(iv) By substituting the values of Q, A, L and ΔT in below Equation, thermal conductivity of the material can be measured in $W/mK \;(or\, W/m\,^oC)$.

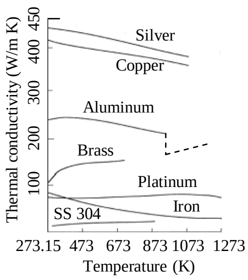
$$k = \frac{Q \times L}{A \times \Delta T} = \frac{W \times m}{m^2 \times K} = \frac{W}{mK}$$

(iii) Thermal conductivity for solid metals and its variation with temperature: Heat conduction in the solid materials takes place due to the migration of free electrons and lattice vibrations which are additive effects. Thus, the thermal conductivity k of a solid may be considered as the sum of the electronic component k_e (i.e., thermal conductivity due to free flow of electrons) and lattice component k_l (i.e., thermal conductivity due to lattice vibrations). That is, $k = k_e + k_l$. Generally, the energy transfer due to electronic component is more effective than lattice component. For a pure solid metal k_e is much larger than k_l, therefore, pure solid metals have the highest thermal conductivity. The values of thermal conductivity in W/mK for some of the solid metals at room temperature (300 K) are given in below Table.

Metals	k (W/mK)	**Metals**	k (W/mK)
Silver	429	Zinc	116

Copper	401	Nickel	90.7
Gold	317	Iron	80.2
Aluminum	237	Cast iron	55-65
Tungsten	174	Stainless steel (302)	15.1

The motion of free electrons in metals at elevated temperatures is obstructed due to increased lattice vibrations. Thus, thermal conductivity of pure metals decreases as the temperature increases (aluminum, mercury and uranium being the exceptions). The variation of thermal conductivity for some metals and alloys are shown in below Figure.



(iv) Thermal conductivity and its variation with temperature for alloys: The presence of any impurity (foreign metal molecules) or alloying of metals causes decrease in thermal conductivity than that of either metal. For example, thermal conductivity of constantan (an alloy of 55% copper and 45% nickel) is $23W/mK$ which is much lower than the thermal conductivity of both the copper and nickel. Thermal conductivity of an alloy increases with increasing temperature.

(v) Thermal conductivity and its variation with temperature for non-metals: In non-metallic solids there is almost no electronic component, thus their thermal conductivity k is primarily determined by k_l. Therefore, non-metals have lower thermal conductivity than metals and are called thermal insulators. However, due to perfect lattice arrangement diamond has the highest known thermal conductivity at room temperature. Thermal conductivity of some of the solid non-metals at room temperature is given in below Table.

Non-metals	k (W/mK)	**Non-metals**	k (W/mK)
Diamond	2300	Asbestos	0.149
Bakelite	1.4	Soft rubber	0.13
Glass	0.78	Cotton	0.06
Brick	0.72	Cork	0.048
Wood	0.17	Glass fiber	0.043

For non-metallic solids, the value of k_l increases with increasing temperature due to larger interactions between the atoms and lattice, thus thermal conductivity increases with rising temperature.

Many building and insulating materials (like brick, concrete, asbestos, foams and fibers) have a porous structure generally filled with air. Since air is a poor heat conductor thus, thermal conductivity of air filled porous insulating materials is low.

(vi) Superconductors: Materials that show very high thermal conductivity at temperature near absolute zero are called superconductors. For example, thermal conductivity of copper at $20\,K$ is of the order of $20000 W/mK$ which is nearly 50 times the conductivity of copper at room temperature.

(vii) Weidemann-Franz law: It states that the ratio of thermal conductivity (k) to the electrical conductivity (σ_e) is the same for all metals at the same temperature and this ratio is directly proportional to the absolute temperature (T) of the metal.

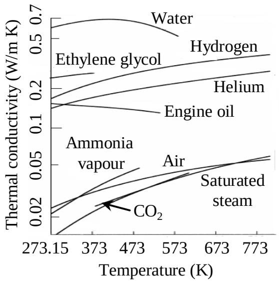
Thus $\frac{k}{\sigma_e} \propto T$ or $\frac{k}{\sigma_e T} = C$

Here, constant C is also called as Lorentz number having a value of $2.45 \times 10^{-8}\, W\Omega/K^2$. This law holds good for a large number of metals between -100 °C and 100 °C and suggests that materials that are good electrical conductors are also good conductors of heat.

(viii) Thermal conductivity for liquids and gases and its variation with temperature: Usually thermal conductivity of liquids lies between those of solids and gases. Liquid metals have higher thermal conductivity than non-metallic liquids and are used in nuclear applications, e.g., sodium is used as coolant in nuclear reactors. The values of thermal conductivity in W/mK for some of the liquids and gases (fluids) at room temperature (300 K) are given in below Tables.

Liquids	k (W/mK)	**Gases**	k (W/mK)
Mercury	8.54	Air	0.026
Water	0.613	Steam	0.0206
Sugarcane juice	0.44	Carbon dioxide	0.0146
Glycerin	0.286	Hydrogen	0.172
Milk (whole)	0.58	Helium	0.073
Ethylene glycol	0.252	Oxygen	0.026

Thermal conductivity of most of the liquids decreases with increase in the temperature (except water and glycerin). It also decreases with increasing molar mass of liquids. Thermal conductivity of a gas is independent of pressure, but it is proportional to the square root of the absolute temperature and inversely proportional to the square root of its molar mass. The variation of thermal conductivity with temperature for some liquids and gases are shown in below Figure.



(ix) Variation of thermal conductivity of different materials: Thermal conductivity of different materials decreases in the order as: Pure crystals, pure metals, alloys, non-metallic solids, liquids and gases.

1.17. Thermal diffusivity: The ratio of the thermal conductivity to the heat capacity is called as thermal diffusivity. It is denoted by α and has the units of m^2/s. Thermal diffusivity is the property of a material which is considered in transient heat conduction analysis. Mathematically, thermal diffusivity can be given as below.

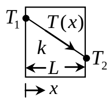
$$\alpha = \frac{\text{Thermal conductivity}}{\text{Heat capacity}} = \frac{k}{\rho c}$$

Here, c is the specific heat of the material in J/kgK that represents the heat storage ability of a material per unit mass and ρ is the density in kg/m^3.

Thermal diffusivity tells about how fast heat diffuses through a material. The larger the value of thermal diffusivity, the faster is the propagation of heat deep into the solid.

Example 1.1 The two sides of the 10 cm thick wall made up of fire brick with $k = 0.72\,W/mK$ are maintained at 1200 K and 900 K, respectively. Determine the rate of heat transfer per unit area through the given wall.

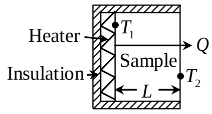
Solution: Refer below Figure. Let $L = 10\,cm = 0.1\,m$, $k = 0.72\,W/mK$, $T_1 = 1200\,K$ and $T_2 = 900\,K$.



$$q = \frac{Q}{A} = k\frac{(T_1 - T_2)}{L} = 0.72 \times \frac{(1200-900)}{0.1} = 2160\,W/m^2$$ **(Ans.)**

Example 1.2. A 25 cm thick rectangular sample of size $0.8\,m \times 0.4\,m$ on a side is used in an experiment. The temperatures of the two surfaces of the sample are 125°C and 115°C. After attaining the steady state the electric heater is observed to consume 0.5 amperes at 125 volts. Determine the thermal conductivity of the given solid sample.

Solution: Refer below Figure. Let $L = 25\,cm = 0.25\,m$, $l = 0.8\,m$, $b = 0.4\,m$, $T_1 = 125^oC$, $T_2 = 115^oC$, $I = 0.5\,amperes$, $V = 125\,volts$.



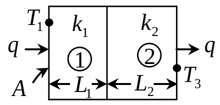
$Q = V \times I = 125 \times 0.5 = 62.5\,W$; $A = l \times b = 0.8 \times 0.4 = 0.32\,m^2$;

$\Delta T = T_1 - T_2 = 125 - 115 = 10^o C$

$$\therefore k = \frac{Q \times L}{A \times \Delta T} = \frac{62.5 \times 0.25}{0.32 \times 10} = 4.883\,W/m\,^oC$$ **(Ans.)**

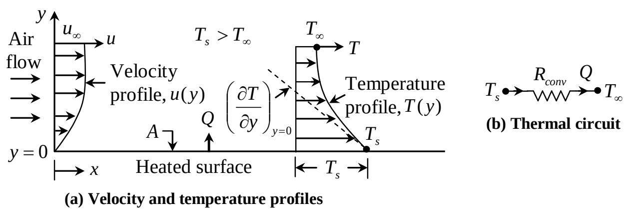
Example 1.3. Two large aluminum plates $(k = 237\,W/mK)$ 2 cm and 3 cm thick are in perfect contact. Determine the value of heat flux when the temperatures at the inner and outer surfaces are 600 K and 580 K.

Solution: Refer below Figure. Let $k_1 = k_2 = 237\,W/mK$, $L_1 = 2\,cm = 0.02\,m$, $L_2 = 3\,cm = 0.03\,m$, $T_1 = 600\,K$ and $T_3 = 580\,K$.



$$q = \frac{T_1 - T_3}{L_1/k_1 + L_2/k_2} = \frac{600 - 580}{(0.02/237) + (0.03/237)} = 94800\,W/m^2$$ **(Ans.)**

1.18. Velocity boundary layer and thermal boundary layer: Heat transfer between a moving fluid and a solid surface maintained at different temperatures occurs due to convection. It happens by the combined effects of conduction within the fluid that is due to random motion of its molecules (diffusion) and the bulk motion of the fluid that removes the heated fluid near the surface and replaces it by colder fluid.



(a) Velocity and temperature profiles

(b) Thermal circuit

Above Figure illustrates the flow of fluid (air) over a heated surface. Because of no slip condition the air in direct contact with the surface sticks to it. Due to this air-surface interaction a region develops in the air through which velocity varies from zero at the surface to the free stream velocity of air (u_∞). This region of air in which the effects of the viscous shearing forces caused by air viscosity are felt is called the velocity boundary layer (δ).

Similarly a region develops in the air through which temperature varies from T_s at the surface to the temperature of the air sufficiently far from the surface, i.e., T_∞. This region is called as thermal boundary layer (δ_t). The air and the heated surface attain the same temperature at the point of contact and this condition is called as no temperature jump condition.

1.19. Newton's law of cooling (basic law of convection): The rate of convection heat transfer (Q) from a heated surface maintained at temperature T_s to the cold fluid at temperature T_∞ flowing over its top surface can be given by Newton's law of cooling as follows.

$$Q = hA(T_s - T_\infty)$$

$$q = \frac{Q}{A} = h(T_s - T_\infty)$$

Here, q is the heat flux to the bulk fluid by convection, h is the convective heat transfer coefficient and A is the surface area through which convective heat transfer occurs.

1.20. Convective thermal resistance and thermal conductance for convection: Newton's law of cooling is given by,

$$Q = hA(T_s - T_\infty)$$

or $$Q = \frac{T_s - T_\infty}{1/(hA)} = \frac{\Delta T}{R_{conv}} = \frac{\text{Temperature difference}}{\text{Convective thermal resistance}}$$

Here, $R_{conv} = 1/(hA)$ is the convective thermal resistance to heat flow.

The thermal conductance for convection, $K_{conv} = 1/R_{conv} = hA$.

1.21. Convective heat transfer coefficient: The convection heat transfer studies ultimately reduce to a study of the means by which convective heat transfer coefficient (h) may be determined. The convective heat transfer coefficient is not a property of the fluid and sometimes it is also called as film conductance or film heat transfer coefficient. The convective heat transfer coefficient is determined experimentally. The units for convective heat transfer coefficient can be given as,

$$h = \frac{Q}{A(T_s - T_\infty)} = \frac{W}{m^2 K}$$

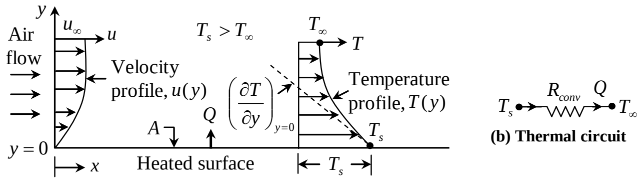
The convective heat transfer coefficient is expressed in W/m^2K (or $W/m^2\,^oC$).

Its value depends on conditions in the boundary layer, which is influenced by the parameters namely, (i) geometry and roughness of the surface, (ii) temperature difference, (iii) bulk fluid velocity, (iv) thermal properties of fluid such as k, c, ρ *and* μ and (v) nature of fluid motion (laminar or turbulent).

1.22. Determination of the convective heat transfer coefficient: In below Figure the fluid layer (air layer) adjacent to the surface is at rest thus, the rate of heat transfer from the surface is by conduction and that can be obtained by applying Fourier's law to the fluid at $y = 0$ as below.

$$q = -k_f \left(\frac{\partial T}{\partial y} \right)_{y=0} \qquad \text{(i)}$$

Here, k_f is the thermal conductivity of the fluid, T is the temperature distribution in the fluid and $(\partial T/\partial y)_{y=0}$ is the temperature gradient at the solid surface.



(a) Velocity and temperature profiles

This heat given by above equation will be convected away from the surface as a result of fluid motion. Therefore, the heat flux given by this equation is also the heat flux to the bulk fluid by convection given by $q = Q/A = h(T_s - T_\infty)$. Thus, by equating these two equations, we get the value of the convective heat transfer coefficient as follows.

$$h(T_s - T_\infty) = -k_f \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$\therefore h = \frac{-k_f \left(\partial T / \partial y\right)_{y=0}}{(T_s - T_\infty)}$$

From this equation the value of convective heat transfer coefficient at a certain position x in flow direction can be determined by measuring the temperature gradient at the surface and the temperature difference between the surface and the fluid.

The thickness of thermal boundary layer (δ_t) increases along the flow direction (x-direction), consequently, temperature gradient $\left(\partial T / \partial y\right)_{y=0}$ in the boundary layer decreases. Therefore, the convective heat transfer coefficient decreases along the flow direction. The average convective heat transfer coefficient can be determined by averaging the local convection heat transfer coefficient over the entire surface of the plate.

Typical values of convective heat transfer coefficient h in W/m^2K encountered in various engineering problems are: (i) Free convection of gases: 2-25, (ii) Forced convection of gases: 25-250, (iii) Free convection of liquids: 10-1000, (iv) Forced convection of liquids: 50-20000 and (v) Boiling and condensation: 2500-100000.

Example 1.4. Cold air at 25°C flows over a hot flat plate maintained at 225°C. If size of the plate is $1\,m \times 0.5\,m$ and the convective heat transfer coefficient is $20\,W/m^2\,^oC$, calculate the heat transfer from the plate. Also calculate the convection thermal resistance.

Solution: Let $T_\infty = 25^oC$, $T_s = 225^oC$, $l = 1\,m$, $b = 0.5\,m$ and $h = 20\,W/m^2\,^oC$.

$$A = l \times b = 1 \times 0.5 = 0.5\,m^2$$

$$Q = hA(T_s - T_\infty) = 20 \times 0.5 \times (225 - 25) = 2000W$$ **(Ans.)**

$$R_{conv} = \frac{1}{hA} = \frac{1}{20 \times 0.5} = 0.1\,^oC/W$$ **(Ans.)**

Example 1.5. The heating element of an immersion water heater of rating 1.5 kW is 20 mm in diameter and 1.6 m in length. (i) Determine the surface temperature of the heating element when it is submerged in water at 25°C and the convective heat transfer coefficient is $400\,W/m^2\,^oC$. (ii)

What will be its surface temperature if by mistake it is used in air at 30°C and the convective heat transfer coefficient is $10\,W/m^2\,^oC$. Give your comment.

Solution: Let $Q = 1.5\,kW = 1500\,W$, $d = 20\,mm = 0.02\,m$, $l = 1.6\,m$, $T_w = 25^oC$, $h_w = 400\,W/m^2\,^oC$, $T_a = 30^oC$ and $h_a = 10\,W/m^2\,^oC$.

$$A = \pi dl = \pi \times 0.02 \times 1.6 = 0.1\,m^2$$

(i) When heater is used in water

$$Q = h_w A(T_s - T_w)$$

$$1500 = 400 \times 0.1 \times (T_s - 25) \Rightarrow \therefore T_s = 62.5\,^oC$$ **(Ans.)**

(i) When heater is used in air

$$Q = h_a A(T_s - T_a)$$

$$1500 = 10 \times 0.1 \times (T_s - 30) \Rightarrow \therefore T_s = 1530\,^oC$$ **(Ans.)**

Such a high surface temperature will lead to the melting of heating element, hence should never be used in air.

Example 1.6. In a flow of liquid at 80°C over a metal surface the temperature profile is found to be $T(y) = (75 + 60y + 0.1y^2)\,^oC$. If the thermal conductivity of the fluid is $0.67\,W/m\,^oC$, evaluate the convective heat transfer coefficient.

Solution: Refer Figure in Question 37. Let $T_\infty = 80^oC$, $T(y) = (75 + 60y + 0.1y^2)\,^oC$ and $k_f = 0.67\,W/m\,^oC$.

$$T(y) = (75 + 60y + 0.1y^2)\,^oC \qquad \text{(i)}$$

$$\frac{\partial T}{\partial y} = (60 + 0.2y)\,^oC/m \qquad \text{(ii)}$$

At $y = 0$, from the expressions (i) and (ii), we obtain the following values.

$$T_s = 75 + 60(0) + 0.1(0^2) = 75\,^oC$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = [60 + 0.2(0)] = 60\,^oC/m$$

$$\therefore h = \frac{-k_f\left(\partial T/\partial y\right)_{y=0}}{(T_s - T_\infty)} = \frac{-0.67 \times 60}{(75 - 80)} = 8.04\,W/m^2\,^oC$$ **(Ans.)**

Example 1.7. A hot plate of area 0.225 m^2 is maintained at a temperature of 333 K by 200 W electric heater when the room temperature is 293 K. Determine the fraction of heat supplied lost by free convection if the convection heat transfer coefficient is given by the relation $h = 4.581(\Delta T)^{1/4}\,W/m^2K$. State what happens to the rest of the heat supplied.

Solution: Let $A = 0.225\,m^2$, $T_s = 333\,K$, $Q_s = 200\,W$, $T_a = 293\,K$ and $h = 4.581(\Delta T)^{1/4}\,W/m^2K$.

$$\Delta T = T_s - T_a = 333 - 293 = 40\,K$$

$$h = 4.581 \times 40^{1/4} = 11.52\,W/m^2K$$

Heat lost by convection can be given as below.

$$Q = hA\Delta T = 11.52 \times 0.225 \times 40 = 103.68\,W$$

Fraction of the heat supplied lost by convection

$$= \frac{Q}{Q_s} \times 100 = \frac{103.68}{200} \times 100 = 51.84\%$$ **(Ans.)**

The remaining (100 - 51.84) = 48.16% is lost by radiation.

1.23. Thermal radiation and blackbody

Thermal radiation: The electromagnetic radiation propagated due to temperature difference is called thermal radiation. In this text book, we limit discussion to thermal radiation that differs from other forms of electromagnetic radiation such as x-rays, microwaves, gamma rays, radio waves and television waves which are not related to temperature.

Blackbody: A black body is an ideal body that absorbs all the incident radiation and reflects or transmits none. It appears black in colour due to the absorption of all visible radiation which occurs in the narrow wavelength range of $0.4\ \mu m$ to $0.76\ \mu m$. At a given temperature and wavelength, no surface can emit more radiation than a blackbody.

1.24. Stefan-Boltzmann law and total emissive power of a black body

Stefan-Boltzmann law: This law states that the rate of radiation energy emitted by a black body per unit surface area is proportional to the fourth power of its absolute temperature.

Let Q be the rate of radiation energy emitted by a black body in $watts\,(W)$, A be its surface area in m^2 and T_s be its surface temperature in Kelvin (K).

Then $\frac{Q}{A} \propto T_s^{\,4}$ or $\frac{Q}{A} = \sigma T_s^{\,4}$

Here, the constant of proportionality, σ is called the Stefan-Boltzmann constant and it has the value of $5.67 \times 10^{-8}\,W/m^2K^4$.

In above equation, $(Q/A) = q_b$ is the flux of heat energy emitted by radiation, i.e., quantity of radiation energy emitted per unit time per unit area by the black body (also called total emissive power of a black body, E_b) in W/m^2.

$$\therefore \frac{Q}{A} = q_b = E_b = \sigma T_s^{\,4}$$

Therefore, the rate at which energy is radiated by a black body at the absolute temperature (T_s) can be given from above equation as,

$$Q = \sigma A T_s^{\ 4}$$

1.25. Emissivity: The rate of radiation energy emitted by all real surfaces will always be less than that emitted by a blackbody at the same temperature and it can be given as below.

$$Q = \varepsilon \sigma A T_s^{\ 4}$$

Here, ε is a radiative property of the surface called as emissivity whose value lies between 0 and 1. For a black body $\varepsilon = 1$. Emissivity provides a measure of how efficiently a real surface emits energy relative to a black body. Mathematically, emissivity can be expressed as below.

$$\varepsilon = \frac{Radiation of\ a\, real\, body\, at\, T_s(K)}{Radiation of\ a\, black\, body\, at\, T_s(K)}$$

Some emissivity data for few surfaces at room temperature (300 K) are: (i) Aluminum foil: 0.07, (ii) Polished stainless steel: 0.17, (iii) Black paint: 0.98, (iv) White paint: 0.9, (v) White paper: 0.92-0.97, (vi) Red brick: 0.93-0.96 and (vii) Water: 0.96.

Example 1.8. Determine the increase in emissive power of a blackbody when it is heated from 25°C to 90 °C.

Solution: Let $T_{s1} = 25^o C = 298.15\, K$ and $T_{s2} = 90^o C = 363.15\, K$.

Using equation: $E_b = \sigma T_s^{\ 4}$, we get: $E_{b1} = \sigma T_{s1}^{\ 4}$ and $E_{b2} = \sigma T_{s2}^{\ 4}$

Increase in emissive power of the blackbody becomes,

$$E_{b2} - E_{b1} = \sigma T_{s2}^{\ 4} - \sigma T_{s1}^{\ 4} = \sigma (T_{s2}^{\ 4} - T_{s1}^{\ 4})$$

$$\therefore (E_{b2} - E_{b1}) = 5.67 \times 10^{-8} \times (363.15^4 - 298.15^4) = 538.07\, W/m^2$$ **(Ans.)**

1.26. Relations for radiation exchange between two bodies: Consider a black body of surface area A_1 at an absolute temperature T_{s1} is completely enclosed by a much larger surface (or black body) at an absolute temperature T_{s2}. The net heat exchange between these two bodies can be expressed as below.

$$Q = \sigma A_1 (T_{s1}^{\ 4} - T_{s2}^{\ 4})$$

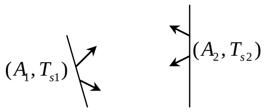
Similarly, the net heat exchange between a real body of surface area A_1, emissivity ε_1, absolute temperature T_{s1} and a black body of surface area A_2 at absolute temperature T_{s2} can be expressed as below.

$$Q = \sigma A_1 \varepsilon_1 (T_{s1}^{\ 4} - T_{s2}^{\ 4})$$

The net heat exchange between two black bodies when one black body of surface area A_1 at absolute temperature T_{s1} faces other black body of surface area A_2 at absolute temperature T_{s2} as shown in below Figure can be given as below.

$$Q = \sigma A_1 F (T_{s1}{}^4 - T_{s2}{}^4)$$

Here, F is called shape factor or view factor (geometric view factor function) that accounts for the geometry and orientation of the two bodies. Shape factor indicates the fraction of radiation leaving the body 1 that strikes body 2 directly.



Radiation heat exchange between two black surfaces

The net heat exchange between two real bodies when one real body of surface area A_1, absolute temperature T_{s1}, emissivity ε_1 faces other real body of surface area A_2, absolute temperature T_{s2}, emissivity ε_2 can be given as below.

$$Q = \sigma A_1 f_\varepsilon F (T_{s1}{}^4 - T_{s2}{}^4) \qquad \text{(i)}$$

Here, f_ε is the equivalent emissivity (emissivity function or interchange factor) for radiant heat exchange between two real bodies. It is pertinent to mention that geometric view factor function F and emissivity function f_ε are not independent of one another as shown in Equation (i).

1.27. Radiation heat transfer coefficient: The radiation heat transfer problems are closely associated with convection heat transfer problems. Therefore, it is convenient to express the rate of radiation heat transfer between a surface of emissivity of ε_1 and area A_1 at temperature T_{s1} and the surrounding surfaces at temperature T_{s2} given by below equation as,

$$Q = \sigma A_1 \varepsilon_1 (T_{s1}{}^4 - T_{s2}{}^4) = h_r A_1 (T_{s1} - T_{s2}) \qquad \text{(i)}$$

Here, h_r is the radiation heat transfer coefficient and can be given as below.

$$h_r = \frac{\sigma A_1 \varepsilon_1 (T_{s1}{}^4 - T_{s2}{}^4)}{A_1 (T_{s1} - T_{s2})} = \sigma \varepsilon_1 (T_{s1}{}^2 + T_{s2}{}^2)(T_{s1} + T_{s2})$$

Equation (i) can also be arranged as,

$$Q = \sigma A_1 \varepsilon_1 (T_{s1}{}^4 - T_{s2}{}^4) = h_r A_1 (T_{s1} - T_{s2}) = \frac{(T_{s1} - T_{s2})}{1/(h_r A_1)} = \frac{(T_{s1} - T_{s2})}{R_{rad}}$$

Here, $R_{rad} = 1/(h_r A_1)$ is the thermal resistance of a surface against radiation and it can be given as below.

$$R_{rad} = \frac{(T_{s1} - T_{s2})}{\sigma A_1 \varepsilon_1 (T_{s1}{}^4 - T_{s2}{}^4)}$$

The thermal conductance for radiation, $K_{rad} = 1/R_{rad} = h_r A_1$.

Example 1.9. If the surface temperature of a polished aluminum with an emissivity of 0.05 exceeds by 15 K than the surrounding at 298 K, determine the radiation heat transfer coefficient.

Solution: Let $\varepsilon_1 = 0.05$, $T_{s1} = (T_{s2} + 15)\,K$ and $T_{s2} = 298\,K$.

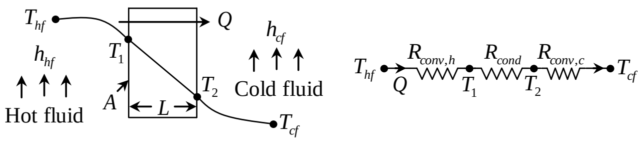
$$T_{s1} = (T_{s2} + 15) = 298 + 15 = 313\,K$$

$$h_r = \sigma\varepsilon_1 (T_{s1}^{\ 2} + T_{s2}^{\ 2})(T_{s1} + T_{s2})$$

$$\therefore h_r = 5.67\times10^{-8} \times 0.05 \times (313^2 + 298^2) \times (313 + 298) = 0.3235\,W/m^2K$$ **(Ans.)**

1.28. Combined heat transfer mechanisms for convection and conduction heat transfer in series: In many practical situations, heat transfer takes place by more than one mechanism in successive steps. The combined heat transfer mechanism for convection and conduction heat transfer in series is described below.

Consider a plane wall of area A and thickness L is heated on one side by a hot fluid and cooled on the other side by a cold fluid as shown in below Figure. Let T_{hf} and h_{hf} be the temperature and convective heat transfer coefficient of hot fluid, respectively, T_{cf} and h_{cf} be the temperature and convective heat transfer coefficient of cold fluid, respectively and T_1 and T_2 be the surface temperatures of the plane wall on hot fluid and cold fluid sides, respectively.



Here, successive heat transfer involves: (i) heat flow from the hot fluid to the wall due to convection, (ii) heat conduction through the wall and (iii) heat transfer from the wall to the cold fluid due to convection. In steady state conditions, the rate of heat transfer through the wall remains constant and it can be given by the following expression.

$$Q = h_{hf} A (T_{hf} - T_1) = kA\frac{T_1 - T_2}{L} = h_{cf} A (T_2 - T_{cf})$$

Form above equation, we get the below expressions.

$$T_{hf} - T_1 = \frac{Q}{h_{hf} A}\ ;\ T_1 - T_2 = \frac{QL}{kA}\ ;\ T_2 - T_{cf} = \frac{Q}{h_{cf} A}$$

By adding these equations, we get:

$$T_{hf} - T_{cf} = Q\left[\frac{1}{h_{hf} A} + \frac{L}{kA} + \frac{1}{h_{cf} A}\right]$$

$$\therefore Q = \frac{T_{hf} - T_{cf}}{[1/h_{hf} A + L/kA + 1/h_{cf} A]}$$

or $$\frac{Q}{A} = q = \frac{T_{hf} - T_{cf}}{[1/h_{hf} + L/k + 1/h_{cf}]}$$

Example 1.10. A wall of a house made from brick with $k = 1.2\, W/mK$ is 25 cm thick and 90 m^2 in area. The temperatures of air outside and inside of the house are 308 K and 293 K, respectively and the respective convective heat transfer coefficients are $10\, W/m^2K$ and $20\, W/m^2K$. Calculate (i) the rate of heat transfer through the wall and (ii) inside and outside wall temperatures.

Solution: Refer Figure given in Question 49. Let $k = 1.2\, W/mK$, $L = 25\, cm = 0.25\, m$, $A = 90\, m^2$, $T_{hf} = 308\, K$, $T_{cf} = 293\, K$, $h_{hf} = 10\, W/m^2K$ and $h_{cf} = 20\, W/m^2K$.

(i) $$Q = \frac{T_{hf} - T_{cf}}{[1/h_{hf}A + L/kA + 1/h_{cf}A]}$$

$$\therefore Q = \frac{308 - 293}{[1/(10 \times 90) + 0.25/(1.2 \times 90) + 1/(20 \times 900]} = 3767.44\, W$$ **(Ans.)**

(ii) In steady state conditions, the rate of heat transfer through the wall remains constant.

Thus for determining outside wall temperature (T_1), we have the following expression.

$$Q = h_{hf} A (T_{hf} - T_1)$$

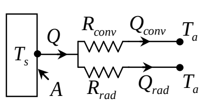
$$3767.44 = 10 \times 90 \times (308 - T_1) \Rightarrow \therefore T_1 = 303.8\, K$$ **(Ans.)**

And for determining inside wall temperature (T_2), we obtain the below expression.

$$Q = h_{cf} A (T_2 - T_{cf})$$

$$3767.44 = 20 \times 90 \times (T_2 - 293) \Rightarrow \therefore T_2 = 295.09\, K$$ **(Ans.)**

1.29. Combined heat transfer mechanisms for convection and radiation heat transfer in parallel: A surface of emissivity ε and area A at very high temperature T_s exposed to the surrounding air at temperature T_a involves convection and radiation simultaneously. The total heat transfer at the surface can be determined by adding the convection and radiation heat transfer components working in parallel as schematically shown in below Figure using electric analogy.



Let h and h_r be the convective and radiative heat transfer coefficients, respectively. The rate of heat transfer from the surface can be expressed as below.

$$Q = Q_{conv} + Q_{rad} = hA(T_s - T_a) + \varepsilon\sigma A(T_s^{\,4} - T_a^{\,4})$$ or
$$Q = hA(T_s - T_a) + h_r A(T_s - T_a)$$

or $$Q = (h + h_r)A(T_s - T_a) = h_{combined} A(T_s - T_a)$$

Here, $h_{combined} = (h + h_r)$ is the combined heat transfer coefficient.

Example 1.11. A steam pipe of diameter 40 cm and length 10 m having a surface temperature of 500 K is placed in air at room temperature of 298 K. If the emissivity of the pipe is 0.86 and the convective heat transfer coefficient is $20\, W/m^2K$, calculate the heat loss from the given pipe.

Solution: Let $d = 40\, cm = 0.04\, m$, $l = 10\, m$, $T_s = 500\, K$, $T_a = 298\, K$, $\varepsilon = 0.86$ and $h = 20\, W/m^2K$.

$$A = \pi dl = \pi \times 0.04 \times 10 = 1.257\, m^2$$

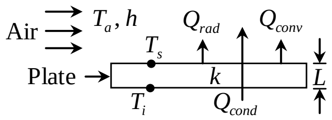
Heat loss due to combined modes of convection and radiation can be given as below.

$$Q = Q_{conv} + Q_{rad} = hA(T_s - T_a) + \varepsilon\sigma A(T_s^{\,4} - T_a^{\,4})$$

$$Q = 20 \times 1.257 \times (500 - 298) + 0.86 \times 5.67 \times 10^{-8} \times 1.257 \times (500^4 - 298^4)$$
$\therefore Q = 8425.77\, W$ **(Ans.)**

Example 1.12. The surface of a horizontal steel plate with $k = 45\, W/m\, ^oC$ measuring 0.9 m long × 0.6 m wide × 0.025 m thick is maintained at a uniform temperature of 305°C and the plate loses 250 W by radiation. If air at 20°C temperature and with convective heat transfer coefficient of $25\, W/m^2\, ^oC$ is blown over the plate, calculate the temperature on inside surface of the plate.

Solution: Let $k = 45\, W/m\, ^oC$, $l = 0.9\, m$, $b = 0.6\, m$, $L = 0.025\, m$, $T_s = 305\, ^oC$, $Q_{rad} = 250\, W$, $T_a = 20\, ^oC$ and $h = 25\, W/m^2\, ^oC$. Let T_i be the temperature on inside surface of the plate as shown in below Figure.



$$A = l \times b = 0.9 \times 0.6 = 0.54\, m^2$$

Now $Q_{cond} = Q_{conv} + Q_{rad}$ (Under steady state conditions)

$$-kA\frac{(T_s - T_i)}{L} = hA(T_s - T_a) + 250$$

$$-45 \times 0.54 \times \frac{(305 - T_i)}{0.025} = 25 \times 0.54 \times (305 - 20) + 250$$

$(305 - T_i) = -4.2 \Rightarrow \therefore T_i = 309.2\, ^oC$ **(Ans.)**

Example 1.13. The surface with $\varepsilon = 0.8$ of a metal plate at 450 K dissipates heat to the air at 308 K by convection and radiation. The thermal conductivity of the plate is $50\, W/mK$ and heat is conducted through the plate to its surface. If the convective heat transfer coefficient is $30\, W/m^2K$, calculate the temperature gradient in the plate.

Solution Refer Figure in Question 54. Let $\varepsilon = 0.8$, $T_s = 450\,K$, $T_a = 308\,K$, $k = 50\,W/mK$ and $h = 30\,W/m^2K$.

$$Q_{cond} = Q_{conv} + Q_{rad} \qquad \text{(Under steady state conditions)}$$

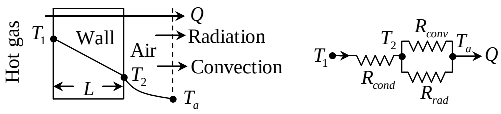
$$-kA\frac{dT}{dx} = hA(T_s - T_a) + \varepsilon\sigma A(T_s{}^4 - T_a{}^4)$$

Thus $$-50\times\frac{dT}{dx} = 30\times(450-308) + 0.8\times5.67\times10^{-8}\times(450^4 - 308^4)$$

$$\therefore \frac{dT}{dx} = -\frac{5711.841}{50} = -114.24\,K/m \quad \textbf{(Ans.)}$$

Example 1.14. The hot combustion gases of furnace are separated from ambient air at 303 K by a 15 cm thick brick wall with $k = 1.2\,W/mK$ and $\varepsilon = 0.8$. Under steady state condition, if the temperature of outer surface is 373 K, the convective heat transfer coefficient of the air adjoining this surface is $20\,W/m^2K$ and area is unity, using electrical analogy calculate the brick inner surface temperature. Take shape factor as unity.

Solution Refer below Figure. Let $T_a = 303\,K$, $L = 15\,cm = 0.15\,m$, $k = 1.2\,W/mK$, $\varepsilon = 0.8$, $T_2 = 373\,K$, $h = 20\,W/m^2K$, $A = 1$ and $F = 1$.



$$R_{cond} = \frac{L}{kA} = \frac{0.15}{1.2\times1} = 0.125\,K/W\,;\quad R_{conv} = \frac{1}{hA} = \frac{1}{20\times1} = 0.05\,K/W\,;$$

$$R_{rad} = \frac{T_2 - T_a}{\sigma A\varepsilon(T_2{}^4 - T_a{}^4)} = \frac{373-303}{5.67\times10^{-8}\times1\times0.8\times(373^4 - 303^4)} = 0.1412\,K/W$$

Since R_{conv} and R_{rad} are in parallel, thus equivalent resistance (R_{eq}) becomes,

$$R_{eq} = \frac{1}{1/R_{conv} + 1/R_{rad}} = \frac{1}{(1/0.05) + (1/0.1412)} = 0.0374\,K/W$$

Now $$Q = \frac{T_1 - T_2}{R_{cond}} = \frac{T_2 - T_a}{R_{eq}} \qquad \text{(Under steady state condition)}$$

Thus $$\frac{T_1 - 373}{0.125} = \frac{373-303}{0.037} \Rightarrow \therefore T_1 = 609.5\,K \quad \textbf{(Ans.)}$$

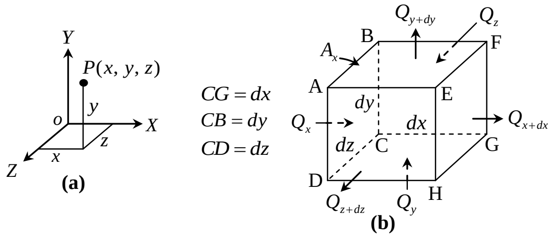
CHAPTER - 2

STEADY STATE HEAT CONDUCTION

Introduction: The specification of the temperature at a point in a material or medium of any geometry can be expressed by choosing a suitable coordinate system such as rectangular (Cartesian), cylindrical, or spherical coordinates. In the analysis of heat transfer, the temperature distribution (the variation of temperature) within the given body is determined by solving differential heat conduction equation subject to boundary conditions of the problem. After determining the temperature distribution, the heat transfer rate at a point within the body in a given direction can be determined by applying the Fourier's law.

In this chapter, the basic equations of heat conduction in rectangular, cylindrical and spherical coordinates have been derived. The heat conduction equation in rectangular coordinate system is used for the analysis of heat transfer in solids with rectangular, square or parallelepiped shapes. The heat conduction equations in cylindrical and spherical coordinates are useful in solving heat conduction problems in cylindrical and spherical shaped bodies, respectively. The boundary conditions required for describing a physical situation in the mathematical form for heat conduction problems are also explained.

2.1. Generalized heat conduction equation in rectangular coordinates: The position of any point P can be located in rectangular coordinates as shown in below Figure (a). Consider the flow of heat through an infinitesimal volume element of sides dx, dy and dz in X, Y and Z directions, respectively in rectangular coordinates as shown in Figure (b). The volume of the element becomes, $dv = dx.dy.dz$. Let k_x, k_y and k_z be the thermal conductivities of the material in the x-, y- and z-directions, respectively and A_x, A_y and A_z be the areas normal to the respective heat flow directions.



The rate of heat flowing into the control volume through the face $ABCD$ can be given by Fourier law as,

$$Q_x = -k_x A_x \frac{\partial T}{\partial x} = -k_x (dydz) \frac{\partial T}{\partial x}$$

The rate of heat flowing out the control volume through the face $EFGH$ is Q_{x+dx} and it can be given by Taylor expansion series (considering the first two terms only and neglecting the higher order terms) as,

$$Q_{x+dx} = Q_x + \frac{\partial}{\partial x}(Q_x)dx$$

The rate of heat stored in the volume element due to conduction of heat in x-direction (dQ_x) can be given as,

$$dQ_x = Q_x - Q_{x+dx} = Q_x - \left[Q_x + \frac{\partial}{\partial x}(Q_x)dx \right] = -\frac{\partial}{\partial x}(Q_x)dx$$

Substituting the value of Q_x, we get:

$$dQ_x = -\frac{\partial}{\partial x}\left[-k_x(dydz)\frac{\partial T}{\partial x} \right]dx = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right)dxdydz = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right)dv$$

Similarly, the rate of heat stored in the volume element due to conduction of heat in y-direction (dQ_y) and z-direction (dQ_z) can be given by the following equations.

$$dQ_y = \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y} \right)dv; \; dQ_z = \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z} \right)dv$$

Net rate of heat stored in the element (Q_{net}) due to conduction of heat in x-, y- and z-directions can be obtained as: $Q_{net} = dQ_x + dQ_y + dQ_z$

Thus $Q_{net} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right)dv + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y} \right)dv + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z} \right)dv$

Let there be constant rate of heat generation per unit volume of the element (q_g) then, the rate of heat stored in the element due to heat generation (Q_g) is given as,

$$Q_g = q_g \times dxdydz = q_g dv$$

The net rate of heat stored into the element from all the three directions and rate of heat stored in the element due to heat generation causes increase in internal energy of this element. Let the temperature of the element increase by dT in time dt. Then, the rate of increase in internal energy of the element (Q_{st}) is given as,

$$Q_{st} = \rho \times dv \times c \times \frac{\partial T}{\partial t}$$

Here, ρ is the density in kg/m^3, c is the specific heat of the material of the element in J/kgK and t is the time in s.

The energy balance for the volume element can be obtained by applying the first law of thermodynamics. According to this law,

$$Q_{net} + Q_g = Q_{st}$$

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right)dv + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y} \right)dv + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z} \right)dv + q_g dv = \rho dvc\frac{\partial T}{\partial t}$$

or $\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z} \right) + q_g = \rho c\frac{\partial T}{\partial t}$

In heat transfer analysis, usually a material is assumed as isotropic for which $k_x = k_y = k_z = k$. Therefore, this equation becomes,

$$k\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)\right]+q_g = \rho c\frac{\partial T}{\partial t}$$

or $$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}=\frac{\rho c}{k}\frac{\partial T}{\partial t}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(i)}$$

In this equation, $\alpha = k/(\rho c)$ is the thermal diffusivity that tells about how fast heat diffuses through a material and its units are m^2/s. This equation is the general heat conduction equation in rectangular (Cartesian) coordinates for heat conduction under unsteady state through a homogeneous and isotropic material. This equation is used to analyse heat transfer in rectangular, square or parallelepiped shaped solid bodies such as walls, cubes, plates and slabs.

Special forms of general heat conduction equation in rectangular coordinates are:

(i) For steady state heat conduction $(\partial T/\partial t)=0$, therefore, Equation (i) becomes,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}=0 \qquad \text{(Poisson equation)}$$

(ii) In the absence of any internal heat generation within the body, $q_g = 0$, therefore, Equation (i) becomes,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(Fourier equation or diffusion equation)}$$

(iii) For steady state heat conduction and no heat generation, Equation (i) becomes,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \qquad \text{(Laplace equation)}$$

(iv) For one dimensional (say along x-direction) steady state heat conduction and no heat generation, Equation (i) becomes,

$$\frac{\partial^2 T}{\partial x^2}=0 \quad \text{or} \quad \frac{d^2 T}{dx^2}=0$$

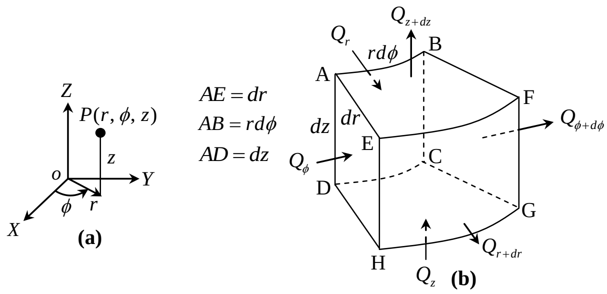
The partial and ordinary derivatives of a function become identical when the function depends on a single variable.

(v) For one dimensional steady state heat conduction with heat generation, Equation (i) becomes,

$$\frac{d^2 T}{dx^2}+\frac{q_g}{k}=0$$

2.2. General heat conduction equation in cylindrical coordinates: The position of any point P can be located in cylindrical coordinates as shown in Figure (a). Consider the flow of heat through an infinitesimal volume element of sides dr, $rd\phi$ and dz in radial, tangential and axial directions, respectively in cylindrical coordinates as shown in Figure (b). The volume of the element is given as, $dv = dr.rd\phi.dz$.

Let thermal conductivity (k), density (ρ) and specific heat (c) of the material do not vary with position and q_g be the constant rate of heat generation per unit volume.



(i) Radial direction (z - ϕ plane): The rate of heat flowing into the control volume through the face $ABCD$ can be given by Fourier law as,

$$Q_r = -k(rd\phi.dz)\frac{\partial T}{\partial r}$$

The rate of heat flowing out the control volume through the face $EFGH$ is Q_{r+dr} and it can be given by Taylor expansion series as (considering the first two terms only),

$$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)dr$$

The rate of heat stored in the volume element due to conduction of heat in radial direction (dQ_r) can be given as,

$$dQ_r = Q_r - Q_{r+dr} = Q_r - \left[Q_r + \frac{\partial}{\partial r}(Q_r)dr\right] = -\frac{\partial}{\partial r}(Q_r)dr$$

Substituting the value of Q_r we get:

$$dQ_r = -\frac{\partial}{\partial r}\left[-k(rd\phi.dz)\frac{\partial T}{\partial r}\right]dr = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dr.d\phi.dz$$

or $$dQ_r = k\left(r\frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{\partial r}\right)dr.d\phi.dz = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)dr.rd\phi.dz$$

or $$dQ_r = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)dv$$

(ii) Tangential direction (r - z plane): The rate of heat flowing into the control volume through the face $AEHD$ can be given by Fourier law as,

$$Q_\phi = -k(dr.dz)\frac{\partial T}{r\partial \phi}$$

The rate of heat flowing out the control volume through the face $BFGC$ is $Q_{\phi+d\phi}$ and it can be given by Taylor expansion series as,

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\partial\phi}(Q_\phi) r d\phi$$

The rate of heat stored in the volume element due to conduction of heat in tangential direction (dQ_ϕ) can be given as,

$$dQ_\phi = Q_\phi - Q_{\phi+d\phi} = Q_\phi - \left[Q_\phi + \frac{\partial}{r\partial\phi}(Q_\phi) r d\phi \right] = -\frac{\partial}{r\partial\phi}(Q_\phi) r d\phi$$

By substituting the value of Q_ϕ we get:

$$dQ_\phi = -\frac{\partial}{r\partial\phi}\left[-k(dr.dz)\frac{\partial T}{r\partial\phi} \right] r d\phi = \frac{k}{r^2}\frac{\partial^2 T}{\partial\phi^2} dr.rd\phi.dz = \frac{k}{r^2}\frac{\partial^2 T}{\partial\phi^2} dv$$

(iii) Axial direction (r - ϕ plane): The rate of heat flowing into the control volume through the face $DCGH$ can be given by Fourier law as,

$$Q_z = -k(rd\phi.dr)\frac{\partial T}{\partial z}$$

The rate of heat flowing out the control volume through the face $ABFE$ is Q_{z+dz} and it can be given by Taylor expansion series as,

$$Q_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q_z) dz$$

The rate of heat stored in the volume element due to conduction of heat in axial direction (dQ_z) can be given as,

$$dQ_z = Q_z - Q_{z+dz} = Q_z - \left[Q_z + \frac{\partial}{\partial z}(Q_z) dz \right] = -\frac{\partial}{\partial z}(Q_z) dz$$

By substituting the value of Q_z, we get:

$$dQ_z = -\frac{\partial}{\partial z}\left[-k(rd\phi.dr)\frac{\partial T}{\partial z} \right] dz = k\frac{\partial^2 T}{\partial z^2} dr.rd\phi.dz = k\frac{\partial^2 T}{\partial z^2} dv$$

Net rate of heat stored in the element (Q_{net}) due to conduction of heat in radial, tangential and axial directions can be obtained as: $Q_{net} = dQ_r + dQ_\phi + dQ_z$

Thus $$Q_{net} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) dv + \frac{k}{r^2}\frac{\partial^2 T}{\partial\phi^2} dv + k\frac{\partial^2 T}{\partial z^2} dv$$

The rate of heat stored in the element due to heat generation (Q_g) is given as,

$$Q_g = q_g \times dr.rd\phi.dz = q_g dv$$

The rate of increase in internal energy of the element (Q_{st}) due to increase in temperature of the element by dT in time dt is given as,

$$Q_{st} = \rho \times dv \times c \times \frac{\partial T}{\partial t}$$

The energy balance for the volume element is given as,

$$Q_{net} + Q_g = Q_{st}$$

Substituting expressions of Q_{net}, Q_g and Q_{st} in above equation, we get:

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)dv + \frac{k}{r^2}\frac{\partial^2 T}{\partial \phi^2}dv + k\frac{\partial^2 T}{\partial z^2}dv + q_g dv = \rho dvc\frac{\partial T}{\partial t}$$

$$k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right)dv + q_g dv = \rho dvc\frac{\partial T}{\partial t}$$

Dividing both sides by kdv, we obtain the below expression.

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(i)}$$

This equation is the general heat conduction equation in cylindrical coordinates which is useful in solving heat conduction problems for cylindrical solid geometries such as rods, tubes and pipes.

Special forms of the general heat conduction equation in cylindrical coordinates are:

(i) When temperature varies in radial direction only, Equation (i) becomes,

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

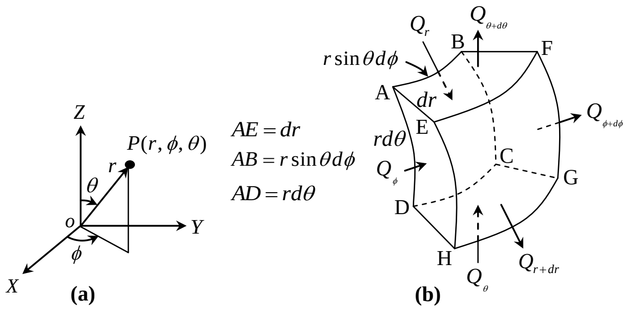
(ii) For steady state one dimensional heat conduction in radial direction and without any heat generation, Equation (i) becomes,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

(iii) For steady state one dimensional heat conduction in radial direction and with heat generation, Equation (i) becomes,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_g}{k} = 0 \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_g}{k}r = 0$$

2.3. General heat conduction equation in spherical coordinates: The position of any point P can be located in spherical coordinates as shown in Figure (a). Consider the flow of heat through an infinitesimal volume element of sides dr, $r\sin\theta d\phi$ and $rd\theta$ in r-, ϕ- and θ-directions, respectively in spherical coordinates as shown in Figure (b). The volume of the element can be given as, $dv = dr.r\sin\theta d\phi.rd\theta$.



Let thermal conductivity (k), density (ρ) and specific heat (c) of the material do not vary with position and q_g be the constant rate of heat generation per unit volume.

(i) θ-ϕ plane; r-direction: The rate of heat flowing into the control volume through the face $ABCD$ can be given by Fourier law as below.

$$Q_r = -k(r\sin\theta d\phi.rd\theta)\frac{\partial T}{\partial r}$$

The rate of heat flowing out the control volume through the face $EFGH$ is Q_{r+dr} and it can be given by Taylor expansion series as,

$$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)dr$$

The rate of heat stored in the volume element due to conduction of heat in r-direction (dQ_r) can be given as,

$$dQ_r = Q_r - Q_{r+dr} = Q_r - \left[Q_r + \frac{\partial}{\partial r}(Q_r)dr\right] = -\frac{\partial}{\partial r}(Q_r)dr$$

Now substituting the value of Q_r, we get:

$$dQ_r = -\frac{\partial}{\partial r}\left[-k(r\sin\theta d\phi.rd\theta)\frac{\partial T}{\partial r}\right]dr = k\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dr.\sin\theta d\phi.d\theta$$

or $$dQ_r = k\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dr.\frac{r\sin\theta d\phi}{r}.\frac{rd\theta}{r} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dr.r\sin\theta d\phi.rd\theta$$

or $$dQ_r = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dv$$

(ii) r-θ plane; ϕ- direction: The rate of heat flowing into the control volume through the face $AEHD$ can be given by Fourier law as below.

$$Q_\phi = -k(dr.rd\theta)\frac{\partial T}{r\sin\theta\partial\phi}$$

The rate of heat flowing out the control volume through the face $BFGC$ is $Q_{\phi+d\phi}$ and it can be given by Taylor expansion as,

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\sin\theta\partial\phi}(Q_\phi)r\sin\theta d\phi$$

The rate of heat stored in the volume element due to conduction of heat in ϕ-direction (dQ_ϕ) can be given as,

$$dQ_\phi = Q_\phi - Q_{\phi+d\phi} = Q_\phi - \left[Q_\phi + \frac{\partial}{r\sin\theta\partial\phi}(Q_\phi)r\sin\theta d\phi\right] = -\frac{\partial}{r\sin\theta\partial\phi}(Q_\phi)r\sin\theta d\phi$$

Now substituting the value of Q_ϕ, we get:

$$dQ_\phi = -\frac{\partial}{r\sin\theta\partial\phi}\left[-k(dr.rd\theta)\frac{\partial T}{r\sin\theta\partial\phi}\right]r\sin\theta d\phi = \frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}dr.r\sin\theta d\phi.rd\theta$$

or $$dQ_\phi = \frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}dv$$

(iii) r-ϕ plane; θ- direction: The rate of heat flowing into the control volume through the face $DCGH$ can be given by Fourier law as below.

$$Q_\theta = -k(r\sin\theta d\phi.dr)\frac{\partial T}{r\partial\theta}$$

The rate of heat flowing out the control volume through the face $ABFE$ is $Q_{\theta+d\theta}$ and it can be given by Taylor expansion as,

$$Q_{\theta+d\theta} = Q_\theta + \frac{\partial}{r\partial\theta}(Q_\theta)rd\theta$$

The rate of heat stored in the volume element due to conduction of heat in θ-direction (dQ_θ) can be given as,

$$dQ_\theta = Q_\theta - Q_{\theta+d\theta} = Q_\theta - \left[Q_\theta + \frac{\partial}{r\partial\theta}(Q_\theta)rd\theta\right] = -\frac{\partial}{r\partial\theta}(Q_\theta)rd\theta$$

Now substituting the value of Q_θ from, we obtain:

$$dQ_\theta = -\frac{\partial}{r\partial\theta}\left[-k(r\sin\theta d\phi.dr)\frac{\partial T}{r\partial\theta}\right]rd\theta = \frac{k}{r}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right]dr.d\phi.rd\theta$$

or $$dQ_\theta = \frac{k}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)dr.\frac{r\sin\theta d\phi}{r\sin\theta}.rd\theta = \frac{k}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)dr.r\sin\theta d\phi.rd\theta$$

or $$dQ_\theta = \frac{k}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)dv$$

Net rate of heat stored in the element (Q_{net}) due to conduction of heat in r -, ϕ - and θ -directions can be obtained as, $Q_{net} = dQ_r + dQ_\phi + dQ_\theta$.

Thus $$Q_{net} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dv + \frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}dv + \frac{k}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)dv$$

The rate of heat stored in the element due to heat generation (Q_g) is given as,

$$Q_g = q_g \times dr.r\sin\theta d\phi.rd\theta = q_g dv$$

The rate of increase in internal energy of the element (Q_{st}) due to increase in temperature of the element by dT in time dt is given by,

$$Q_{st} = \rho \times dv \times c \times \frac{\partial T}{\partial t}$$

The energy balance for the volume element can be given as below.

$$Q_{net} + Q_g = Q_{st}$$

Substituting expressions of Q_{net}, Q_g and Q_{st} in above equation, we get:

$$\frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dv + \frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}dv + \frac{k}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)dv + q_g dv = \rho dvc\frac{\partial T}{\partial t}$$

$$k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right]dv + q_g dv = \rho dvc\frac{\partial T}{\partial t}$$

Dividing both sides by kdv, we obtain the below expression.

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right] + \frac{q_g}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

or $$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right] + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(i)}$$

This is the general heat conduction equation in spherical coordinates which is useful in analysing heat transfer in spherical shaped solid bodies such as spherical metal balls and walls of spherical containers.

Special forms of the general heat conduction equation in spherical coordinates are:

(i) For steady state one dimensional heat conduction in radial direction (r-direction) and in the absence of any heat generation, Equation (i) reduces to the following form.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

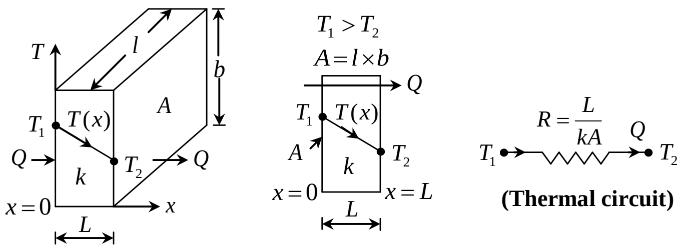
(ii) For steady state one dimensional heat conduction in radial direction (r -direction) and with heat generation, Equation (i) reduces to the following form.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{q_g}{k}=0$$

or $$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{q_g}{k}r^2=0$$

Question 4. For one dimensional steady state heat conduction derive expressions for temperature distribution and rate of heat conduction through a plane wall with constant thermal conductivity and without internal heat generation.

Answer: Consider steady heat conduction through a plane wall (or slab) of thickness L. Let A be its constant cross-sectional area normal to heat flow direction, k be the constant thermal conductivity and there is no heat generation with in this wall. The two faces of the wall are maintained at uniform temperatures T_1 at $x=0$ and T_2 at $x=L$ such that $T_1 > T_2$ as shown in below Figure.



(3-D view of wall) (Front view) (Thermal circuit)

The general heat conduction equation in Cartesian coordinates is given as,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Since (i) $\partial^2 T/\partial y^2=0$ and $\partial^2 T/\partial z^2=0$ (1-D heat transfer), (ii) $\partial T/\partial t=0$ (Steady heat transfer) and (iii) $q_g=0$ (No heat generation), therefore, we get:

$$\frac{d^2T}{dx^2}=0$$

Integrating, we get: $\frac{dT}{dx}=C_1$

Integrating again, we get: $T=C_1x+C_2$ (i)

Here, C_1 and C_2 are arbitrary constants whose values can be obtained with the help of the boundary conditions: (i) $T=T_1$ at $x=0$ and (ii) $T=T_2$ at $x=L$

Substituting first boundary condition (b.c.) in Equation (i), we get:

$$T_1=C_1(0)+C_2 \Rightarrow C_2=T_1$$

Substituting second boundary condition and value of C_2 in Equation (i), we get:

$$T_2 = C_1(L) + T_1 \Rightarrow \therefore C_1 = \frac{T_2 - T_1}{L}$$

Temperature distribution in the wall can be obtained by substituting the values of C_1 and C_2 in Equation (i) as ,

$$T = \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

Differentiating above equation with respect to x , we get:

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

The rate of heat conduction through the wall is given by Fourier's law as,

$$Q = -kA\frac{dT}{dx} = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L}$$

or $$Q = kA\frac{\Delta T}{L} = \frac{\Delta T}{L/(kA)} = \frac{\Delta T}{R} = \frac{\text{Temperature difference}}{\text{Conductive thermal resistance}}$$

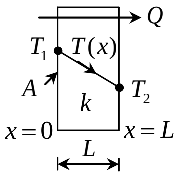
The term $L/(kA)$ in above Equation is called the conductive thermal resistance or simply thermal resistance $(R_{cond}\ or\ R)$. The thermal circuit is also shown in above Figure.

Alternatively:

Rate of heat conduction	Temperature distribution
Fourier equation: $Q = -kA\dfrac{dT}{dx}$ Rearranging and integrating between the limits $T = T_1$ at $x = 0$ and $T = T_2$ at $x = L$. $Q\int_0^L dx = -kA\int_{T_1}^{T_2} dT$ $QL = kA(T_1 - T_2)$ $\therefore Q = \dfrac{kA(T_1 - T_2)}{L}$ (i)	Rearranging & integrating the Fourier equation with in the limits $T = T_1$ at $x = 0$ and T at x as, $Q\int_0^x dx = -kA\int_{T_1}^{T} dT$ or $Qx = kA(T_1 - T)$ $\therefore Q = \dfrac{kA(T_1 - T)}{x}$ (ii) Comparing Equations (i) & (ii), we get: $\dfrac{kA(T_1 - T)}{x} = \dfrac{kA(T_1 - T_2)}{L}$ $\therefore T = \left(\dfrac{T_2 - T_1}{L}\right)x + T_1$

Question 5. The outer and inner sides of the 30 cm thick wall ($k = 0.8\,W/mK$) are maintained at 310 K and 290 K, respectively. Determine the steady rate of heat transfer through the given wall if its length is 4 m and height is 3 m. Also calculate the temperature at an interior point of the wall at 0.2 m distance from the outer surface.

Solution Refer below Figure. Let $L = 30\,cm = 0.3\,m$, $k = 0.8\,W/mK$, $T_1 = 310\,K$, $T_2 = 290\,K$, $l = 4\,m$, $b = 3\,m$ and $x = 0.2\,m$.



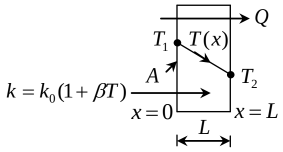
$$A = l \times b = 4 \times 3 = 12\,m^2$$

$$Q = kA\frac{(T_1 - T_2)}{L} = 0.8 \times 12 \times \frac{310 - 290}{0.3} = 640\,W \textbf{ (Ans.)}$$

$$T = \left(\frac{T_2 - T_1}{L}\right)x + T_1 = \left(\frac{290 - 310}{0.3}\right) \times 0.2 + 310 = 296.7\,K \textbf{ (Ans.)}$$

Question 6. Derive expression for rate of heat conduction through a plane wall with variable thermal conductivity.

Answer: Thermal conductivity of many materials can be approximated as, $k = k_0(1 + \beta T)$. Here, k_0 is the thermal conductivity at a reference temperature, β is an empirical constant also called as temperature coefficient of thermal conductivity and T is any temperature.



The rate of heat conduction through a plane wall with variable thermal conductivity can be obtained by substituting the value of $k = k_0(1+\beta T)$ in Fourier Law as,

$$Q = -k_0(1+\beta T)A\frac{dT}{dx}$$

Rearranging the above expression and integrating both sides within the boundary conditions (i) $T = T_1$ at $x = 0$ and (ii) $T = T_2$ at $x = L$ as shown in above Figure.

$$Q\int_0^L dx = -k_0 A\int_{T_1}^{T_2}(1+\beta T)dT \Rightarrow Q[L]_0^L = -k_0 A\left[T + \beta\frac{T^2}{2}\right]_{T_1}^{T_2}$$

$$Q = \frac{k_0 A}{L}\left[\left(T_1 + \beta\frac{T_1^2}{2}\right) - \left(T_2 + \beta\frac{T_2^2}{2}\right)\right] \Rightarrow Q = \frac{k_0 A}{L}\left[(T_1 - T_2) + \beta\frac{(T_1^2 - T_2^2)}{2}\right]$$

$$Q = k_0\left[1 + \beta\frac{(T_1 + T_2)}{2}\right]\frac{A(T_1 - T_2)}{L} \Rightarrow Q = k_0(1+\beta T_m)\frac{A(T_1 - T_2)}{L} = \frac{k_m A(T_1 - T_2)}{L}$$

Here, $k_m = k_0(1+\beta T_m)$ is the mean thermal conductivity evaluated at the arithmetic mean temperature, $T_m = (T_1 + T_2)/2$.

Question 7. The temperatures of the two surfaces of a 0.25 m thick plane fireclay wall are 1250 K and 250 K. The thermal conductivity of the fireclay varies with temperature T in K as per the linear relation, $k(W/mK) = 0.835\times(1+0.00072T)$. Determine (i) the steady heat loss through the wall which is $2\,m\times 0.8\,m$ on a side and (ii) heat flux.

Solution: Let $L = 0.25\,m$, $T_1 = 1250\,K$, $T_2 = 250\,K$, $k = 0.835\times(1+0.00072T)$, $l = 2\,m$ and $b = 0.8\,m$.

(i) $$T_m = \frac{T_1 + T_2}{2} = \frac{1250 + 250}{2} = 750\,K$$

$$k_m = k_0(1+\beta T_m) = 0.835\times(1+0.00072\times 750) = 1.2859\,W/mK$$

$$A = l\times b = 2\times 0.8 = 1.6\,m^2$$

$$Q = \frac{k_m A(T_1 - T_2)}{L} = \frac{1.2859\times 1.6\times(1250-250)}{0.25} = 8229.76\,W$$ **(Ans.)**

(ii) $$q = \frac{Q}{A} = \frac{8229.76}{1.6} = 5143.6\,W/m^2$$ **(Ans.)**

Question 8. Derive expression for rate of heat conduction through a composite wall under one dimensional (1-D) steady state heat conduction and without internal heat generation. Also find the expression for overall heat transfer coefficient.

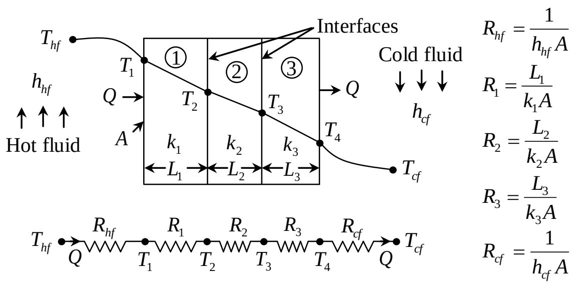
Answer: Consider a composite wall consisting of three layers 1, 2 and 3 having a perfect contact at the interfaces. This wall is heated on left side by a hot fluid and cooled on the right side by a cold fluid as shown in below Figure. Here, steady one-dimension (1-D) heat conduction occurs through the composite wall without any heat generation.

Let L_1, L_2 and L_3 be thicknesses and k_1, k_2 and k_3 be the constant thermal conductivities of the three layers 1, 2 and 3, respectively,

A be the area of the composite wall normal to heat flow direction,

T_{hf} and T_{cf} be the temperatures and h_{hf} and h_{cf} be the convective heat transfer coefficients of hot and cold fluids, respectively,

T_1 and T_4 be the surface temperatures of the composite wall on hot fluid and cold fluid sides, respectively, and T_2 and T_3 be the temperatures at the interfaces.



In steady state conditions, the rate of heat transfer through the wall remains constant and it can be given by the following expression.

$$Q = h_{hf}A(T_{hf} - T_1) = k_1A\frac{T_1 - T_2}{L_1} = k_2A\frac{T_2 - T_3}{L_2} = k_3A\frac{T_3 - T_4}{L_3} = h_{cf}A(T_4 - T_{cf})$$

or $$Q = \frac{T_{hf} - T_1}{1/h_{hf}A} = \frac{T_1 - T_2}{L_1/k_1A} = \frac{T_2 - T_3}{L_2/k_2A} = \frac{T_3 - T_4}{L_3/k_3A} = \frac{T_4 - T_{cf}}{1/h_{cf}A}$$

Thermal resistances are given as:

$$R_{hf} = 1/h_{hf}A;\ R_1 = L_1/k_1A;\ R_2 = L_2/k_2A;\ R_3 = L_3/k_3A;\ R_{cf} = 1/h_{cf}A$$

Thus $$Q = \frac{T_{hf} - T_1}{R_{hf}} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_3}{R_2} = \frac{T_3 - T_4}{R_3} = \frac{T_4 - T_{cf}}{R_{cf}}$$

From above Equation, we get:

$$T_{hf} - T_1 = QR_{hf};\ T_1 - T_2 = QR_1;\ T_2 - T_3 = QR_2;\ T_3 - T_4 = QR_3;\ T_4 - T_{cf} = QR_{cf}$$

Adding these expressions, we get: $T_{hf} - T_{cf} = Q[R_{hf} + R_1 + R_2 + R_3 + R_{cf}]$

$$\therefore Q = \frac{T_{hf} - T_{cf}}{R_{hf} + R_1 + R_2 + R_3 + R_{cf}} = \frac{\Delta T}{\sum R} \qquad \text{(i)}$$

Equation (i) can also be written in terms of overall heat transfer coefficient (U) as,

$$Q = UA(T_{hf} - T_{cf}) = UA\Delta T \qquad \text{(ii)}$$

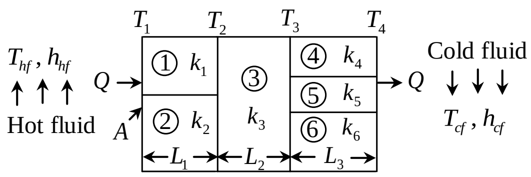
By comparing Equations (i) and (ii), we get: $U = \dfrac{1}{A\sum R}$

Or $$U = \frac{1}{A[1/h_{hf}A + L_1/k_1A + L_2/k_2A + L_3/k_3A + 1/h_{cf}A]}$$

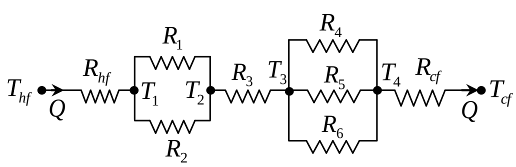
$$\therefore U = \frac{1}{[1/h_{hf} + L_1/k_1 + L_2/k_2 + L_3/k_3 + 1/h_{cf}]}$$

The overall heat transfer coefficient (U) is widely applied in the analysis of the problems related with composite walls and design of heat exchangers.

Question 9. Find the values of total thermal resistance and rate of heat conduction for the composite wall shown in below Figure using electrical analogy.



Answer: Equivalent thermal resistance circuit for the given composite wall becomes,



Here, $R_{hf} = 1/h_{hf}A$, $R_1 = L_1/k_1A_1$, $R_2 = L_1/k_2A_2$, $R_3 = L_2/k_3A_3$, $R_4 = L_3/k_4A_4$, $R_5 = L_3/k_5A_5$, $R_6 = L_3/k_6A_6$ and $R_{cf} = 1/h_{cf}A$.

The equivalent thermal resistances $(R_{eq})_1$ and $(R_{eq})_2$ for conduction through the composite wall containing thermal resistances in parallel can be evaluated as below.

$$\frac{1}{(R_{eq})_1} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \therefore (R_{eq})_1 = \frac{1}{1/R_1 + 1/R_2}$$

$$\frac{1}{(R_{eq})_2} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} \Rightarrow \therefore (R_{eq})_2 = \frac{1}{1/R_3 + 1/R_4 + 1/R_5}$$

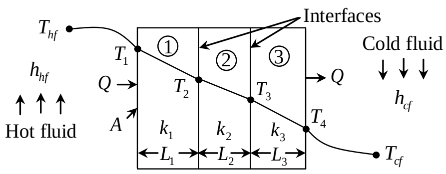
Total thermal resistance,

$$\sum R = R_{hf} + (R_{eq})_1 + R_3 + (R_{eq})_2 + R_{cf}$$

$$\therefore Q = \frac{T_{hf} - T_{cf}}{R_{hf} + (R_{eq})_1 + R_3 + (R_{eq})_2 + R_{cf}} = \frac{\Delta T}{\sum R}$$

Question 10. A steam boiler furnace is made of three layers of thicknesses 25 cm, 10 cm and 15 cm with thermal conductivities $1.64\, W/mK$, k and $9\, W/mK$, respectively. The hot gas temperature is 1520 K having a convection coefficient of $25 W/m^2K$ and the wall temperature inside the furnace is 1375 K. Determine the unknown thermal conductivity if the outside wall is exposed to air at 297 K and the convection coefficient is $12\, W/m^2K$.

Solution: Refer below Figure. Let $L_1 = 25\, cm = 0.25\, m$, $L_2 = 10\, cm = 0.1\, m$, $L_3 = 15\, cm = 0.15\, m$, $k_1 = 1.64\, W/mK$, $k_2 = k$, $k_3 = 9\, W/mK$, $T_{hf} = 1520\, K$, $h_{hf} = 25\, W/m^2K$, $T_1 = 1375\, K$, $T_{cf} = 297\, K$ and $h_{cf} = 12\, W/m^2K$.



Heat flux under steady state can be given as below.

$$q = \frac{Q}{A} = h_{hf}(T_{hf} - T_1) = \frac{T_1 - T_{cf}}{L_1/k_1 + L_2/k_2 + L_3/k_3 + 1/h_{cf}}$$

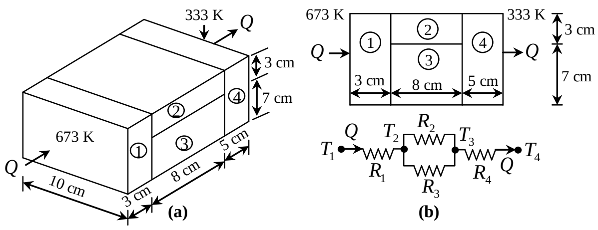
Thus $$25 \times (1520 - 1375) = \frac{1375 - 297}{(0.25/1.64) + (0.1/k) + (0.15/9) + (1/12)}$$

$$3625 \times \left(0.2524 + \frac{0.1}{k}\right) = 1078$$

$$\therefore k = 2.22\, W/mK$$ **(Ans.)**

Question 11. Find the steady 1-D heat flow rate through the composite wall shown in below Figure. Take $k_1 = 150\, W/mK$, $k_2 = 30\, W/mK$, $k_3 = 65\, W/mK$ and $k_4 = 50\, W/mK$.

Solution: Refer below Figure (a). Let $L_1 = 3\, cm = 0.03\, m$, $L_2 = L_3 = 8\, cm = 0.08\, m$, $L_4 = 5\, cm = 0.05\, m$, $k_1 = 150\, W/mK$, $k_2 = 30\, W/mK$, $k_3 = 65\, W/mK$ and $k_4 = 50\, W/mK$. The thermal circuit for the composite wall is shown in Figure (b).



$$R_1 = \frac{L_1}{k_1 A_1} = \frac{0.03}{150 \times (0.1 \times 0.1)} = 0.02\, K/W \; ; \; R_2 = \frac{L_2}{k_2 A_2} = \frac{0.08}{30 \times (0.1 \times 0.03)} = 0.89\, K/W$$

$$R_3 = \frac{L_3}{k_3 A_3} = \frac{0.08}{65 \times (0.1 \times 0.07)} = 0.176\, K/W \; ; \; R_4 = \frac{L_4}{k_4 A_4} = \frac{0.05}{50 \times (0.1 \times 0.1)} = 0.1\, K/W$$

Resistances R_2 and R_3 are in parallel, thus equivalent resistance (R_{eq}) becomes,

$$R_{eq} = \frac{1}{1/R_2 + 1/R_3} = \frac{1}{(1/0.89) + (1/0.176)} = 0.147\, K/W$$

The total thermal resistance can be obtained as below.

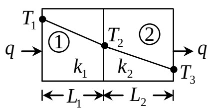
$$\sum R = R_1 + R_{eq} + R_4 = 0.02 + 0.147 + 0.1 = 0.267\, K/W$$

The steady heat flow rate through the composite wall can be evaluated as,

$$Q = \frac{\Delta T}{\sum R} = \frac{673 - 333}{0.267} = 1273.41\, W$$ **(Ans.)**

Question 12. The thickness of a furnace wall made up of firebricks $(k = 0.8\, W/mK)$ and insulation $(k = 0.12\, W/mK)$ is 30 cm. Evaluate the thickness of each layer and the rate of heat loss per unit area of the wall, if under steady state condition the furnace sidewall temperature is 1580 K, ambient side wall temperature is 302 K and the intermediate temperature is 1420 K.

Solution: Refer below Figure. Let $k_1 = 0.8\, W/mK$, $k_2 = 0.12\, W/mK$ $L = L_1 + L_2 = 30\, cm = 0.3\, m$, $T_1 = 1580\, K$, $T_3 = 302\, K$, and $T_2 = 1420\, K$. Let L_1 be the thickness of firebricks, thus $L_2 = (0.3 - L_1)$ be the thickness of the insulation.



Since $q = \frac{T_1 - T_2}{L_1/k_1} = \frac{T_2 - T_3}{L_2/k_2}$ or $\frac{1580 - 1420}{L_1/0.8} = \frac{1420 - 302}{(0.3 - L_1)/0.12}$

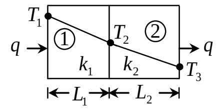
$\therefore L_1 = 0.1465\, m$ **(Ans.)**

$L_2 = 0.3 - L_1 = 0.3 - 0.1465 = 0.1535\, m$ **(Ans.)**

$$q = \frac{T_1 - T_2}{L_1/k_1} = \frac{1580 - 1420}{0.1465/0.8} = 873.72\, W/m^2$$ **(Ans.)**

Question 13. A 60 cm thick wall $(k = 1.2\, W/mK)$ of a furnace is to be insulated with a material $(k = 0.2\, W/mK)$ so that steady heat loss per square meter would not exceed 1200 W. Determine the thickness of insulation if the inner and outer surface temperatures are 1470 K and 300 K, respectively.

Solution: Refer below Figure. Let $L_1 = 60\,cm = 0.6\,m$, $k_1 = 1.2\,W/mK$, $k_2 = 0.2\,W/mK$, $q = 1200 W/m^2$, $T_1 = 1470\,K$ and $T_3 = 300\,K$. Let L_2 be the thickness of the insulation.



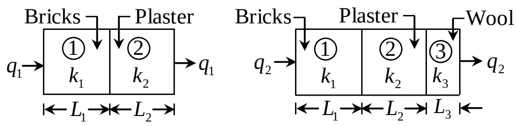
$$q = \frac{T_1 - T_3}{L_1/k_1 + L_2/k_2} \quad \text{or} \quad 1200 = \frac{1470 - 300}{(0.6/1.2) + (L_2/0.2)}$$

$\therefore L_2 = 0.095\,m \; or \; 9.5\,cm$ **(Ans.)**

Question 14. An exterior wall of a house is made up of 10 cm layer of common bricks $(k = 0.72\,W/mK)$ followed by a 4.2 cm layer of gypsum plaster $(k = 0.46\,W/mK)$. Evaluate the thickness of loosely packed rock wool insulation $(k = 0.06\,W/mK)$ to be added to reduce the steady heat loss through the wall by 85%.

Solution Refer below Figure. Let $L_1 = 10\,cm = 0.1\,m$, $k_1 = 0.72\,W/mK$, $L_2 = 4.2\,cm = 0.042\,m$, $k_2 = 0.46\,W/mK$, $k_3 = 0.06\,W/mK$ and $q_2 = 0.15\,q_1$.

Let L_3 be the thickness of rock wool insulation and ΔT be the temperature difference across the composite wall that remains the same in both the cases.



Heat flux under steady state for the composite wall of brick and plaster is given as,

$$q_1 = \frac{\Delta T}{L_1/k_1 + L_2/k_2} = \frac{\Delta T}{(0.1/0.72) + (0.042/0.46)} = 4.3442\Delta T$$

Since addition of rock wool insulation reduces heat loss by 85%, therefore, heat flux will become 15% of q_1, i.e., $q_2 = 0.15\,q_1 = 0.15 \times 4.3442\Delta T = 0.65163\Delta T$.

Now $\quad q_2 = \dfrac{\Delta T}{L_1/k_1 + L_2/k_2 + L_3/k_3}$

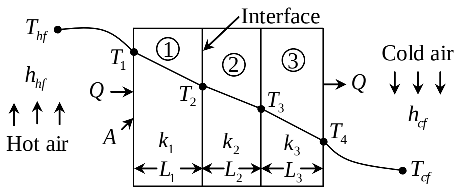
Thus $\quad 0.65163\Delta T = \dfrac{\Delta T}{(0.1/0.72) + (0.042/0.46) + (L_3/0.06)}$

$\therefore L_3 = 0.078\,m \; or \; 7.8\,cm$ **(Ans.)**

Question 15. The composite wall of a room made of 28 cm thick common bricks $(k = 0.7\,W/mK)$ on the inside, followed by a 3.2 cm thick mortar layer $(k = 0.64\,W/mK)$ and 12 cm thick of limestone layer $(k = 0.6\,W/mK)$. The inside and outside air temperatures are 300 K and 268 K, respectively and the corresponding convective heat transfer coefficients are $h_{hf} = 5\,W/m^2K$ and

$h_{cf} = 10\,W/m^2K$. Evaluate (i) overall heat transfer coefficient, (ii) steady rate of heat transfer per unit area and (iii) temperature at the interface of bricks and mortar.

Solution: Refer below Figure. Let $L_1 = 28\,cm = 0.28\,m$, $k_1 = 0.7\,W/mK$, $L_2 = 3.2\,cm = 0.032\,m$, $k_2 = 0.64\,W/mK$, $L_3 = 12\,cm = 0.12\,m$, $k_3 = 0.6\,W/mK$, $T_{hf} = 300\,K$, $T_{cf} = 268\,K$, $h_{hf} = 5\,W/m^2K$ and $h_{cf} = 10\,W/m^2K$.



$$1/h_{hf} = 1/5 = 0.2\,m^2K/W\,;\; L_1/k_1 = 0.28/0.7 = 0.4\,m^2K/W$$

$$L_2/k_2 = 0.032/0.64 = 0.05\,m^2K/W\,;\; L_3/k_3 = 0.12/0.6 = 0.2\,m^2K/W$$

$$1/h_{cf} = 1/10 = 0.1\,m^2K/W$$

(i) The overall heat transfer coefficient can be obtained as,

$$U = \frac{1}{1/h_{hf} + L_1/k_1 + L_2/k_2 + L_3/k_3 + 1/h_{cf}} = \frac{1}{0.2+0.4+0.05+0.2+0.1} = 1.053\,W/m^2K$$

(ii) Steady rate of heat transfer per unit area can be calculated as,

$$q = U(T_{hf} - T_{cf}) = 1.053\times(300-268) = 33.696\,W/m^2 \textbf{ (Ans.)}$$

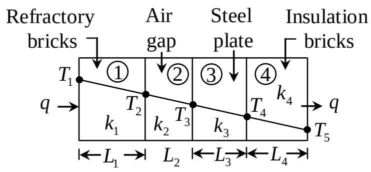
(iii) Temperature at the interface of bricks and mortar can be evaluated as,

$$q = \frac{T_{hf} - T_2}{1/h_{hf} + L_1/k_1} \text{ or } 33.696 = \frac{300 - T_2}{0.2+0.4}$$

$$\therefore T_2 = 300 - 33.696\times 0.6 = 279.78\,K \textbf{ (Ans.)}$$

Question 16. A furnace wall consists of 200 mm layer of refractory bricks $(k = 1.52\,W/mK)$, 6 mm layer of steel plate $(k = 45\,W/mK)$ and 100 mm layer of insulation bricks $(k = 0.138\,W/mK)$. The maximum temperature of the wall is 1423 K on the furnace side and the minimum temperature is 313 K on the outermost side of the wall. If the steady heat loss from the wall is $400\,W/m^2$ and there is air gap between the layers of refractory bricks and steel plate, find how many mm of insulation brick is the air gap equivalent? Also determine the temperature of outer layer surface of the steel plate.

Solution: Refer below Figure. Let $L_1 = 200\,mm = 0.2\,m$, $k_1 = 1.52\,W/mK$, $L_3 = 6\,mm = 0.006\,m$, $k_3 = 45\,W/mK$, $L_4 = 100\,mm = 0.1\,m$, $k_4 = 0.138\,W/mK$, $T_1 = 1423\,K$, $T_5 = 313\,K$, $q = 400\,W/m^2$, Air gap: $L_2 = x\,mm = 0.001x\,m$ and $k_2 = k_4$.



$$L_1/k_1 = 0.2/1.52 = 0.1316\, m^2K/W\ ;\ L_2/k_2 = 0.001x/0.138 = 0.0072x\ m^2K/W$$

$$L_3/k_3 = 0.006/45 = 0.00013 m^2K/W\ ;\ L_4/k_4 = 0.1/0.138 = 0.7246 m^2K/W$$

Since $q = \dfrac{T_1 - T_5}{L_1/k_1 + L_2/k_2 + L_3/k_3 + L_4/k_4}$

Thus $400 = \dfrac{1423 - 313}{0.1316 + 0.0072x + 0.00013 + 0.7246}$

$\therefore x = 266.48\, mm$ **(Ans.)**

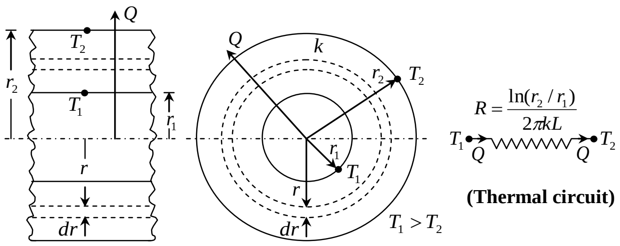
Temperature of outer layer surface of the steel plate can be evaluated as below.

$q = \dfrac{T_4 - T_5}{L_4/k_4}$ or $400 = \dfrac{T_4 - 313}{0.7246}$

$\therefore T_4 = 400 \times 0.7246 + 313 = 602.84\, K$ **(Ans.)**

Question 17. For one dimensional steady state heat conduction derive expressions for temperature distribution and rate of heat conduction through a hollow cylinder with constant thermal conductivity and without internal heat generation.

Answer: Consider steady state one dimensional heat conduction through a hollow cylinder of length L whose both ends are insulated. Let r_1 be the inner radius, r_2 be the outer radius, T_1 and T_2 be the constant temperatures at inner and outer surfaces, respectively, such that $T_1 > T_2$, k be the constant thermal conductivity and there is no heat generation (Refer below Figure). Here, heat flows only in radial direction. Thus temperature is only a function of r-direction i.e., temperature distribution is $T(r)$.



The general heat conduction equation in cylindrical coordinates is given as,

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Since (i) $\partial^2 T/\partial \phi^2 = 0$ and $\partial^2 T/\partial z^2 = 0$ (1-D heat transfer in radial direction only), (ii) $\partial T/\partial t = 0$ (Steady heat transfer) and (iii) $q_g = 0$ (No heat generation), therefore, we get:

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}=0$$

or $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \qquad [\because (1/r)\neq 0]$$

Integrating, we get: $r\frac{dT}{dr}=C_1$ or $\frac{dT}{dr}=\frac{C_1}{r}$

Again integrating, we get: $T=C_1\ln r+C_2$ (i)

Here, C_1 and C_2 are arbitrary constants whose values can be obtained with the help of the two boundary conditions: (i) $T=T_1$ at $r=r_1$ and (ii) $T=T_2$ at $r=r_2$.

Applying first boundary condition, i.e., at $r=r_1$, $T=T_1$, we get:

$$T_1=C_1\ln r_1+C_2 \qquad \text{(ii)}$$

Applying second boundary condition, i.e., at $r=r_2$, $T=T_2$, we get:

$$T_2=C_1\ln r_2+C_2 \qquad \text{(iii)}$$

From the expressions (ii) and (iii), we obtain the values of C_1 and C_2 as,

$$C_1=\frac{T_1-T_2}{\ln(r_1/r_2)}=\frac{T_2-T_1}{\ln(r_2/r_1)} \text{ and } C_2=T_1-C_1\ln r_1=T_1-\frac{T_2-T_1}{\ln(r_2/r_1)}\ln r_1$$

Temperature distribution can be obtained by substituting the values of C_1 and C_2 in Equation (i) as,

$$T=\frac{T_2-T_1}{\ln(r_2/r_1)}\ln r+T_1-\frac{T_2-T_1}{\ln(r_2/r_1)}\ln r_1 \qquad \text{(iv)}$$

Differentiating above Equation with respect to r, we get: $\frac{dT}{dr}=\frac{T_2-T_1}{r\ln(r_2/r_1)}$

Thus, the rate of heat conduction through the cylinder is given by Fourier's law as below.

$$Q=-kA\frac{dT}{dr}=-k(2\pi rL)\frac{T_2-T_1}{r\ln(r_2/r_1)}=2\pi kL\frac{T_1-T_2}{\ln(r_2/r_1)}=\frac{T_1-T_2}{\frac{\ln(r_2/r_1)}{2\pi kL}}=\frac{\Delta T}{R}$$

The term $\frac{\ln(r_2/r_1)}{2\pi kL}$ in above Equation is called the conductive thermal resistance or thermal resistance $(R_{cond}\ or\ R)$ for hollow cylinder. The thermal circuit is also shown in above Figure.

Alternatively: $Q=-kA\frac{dT}{dr}=-k(2\pi rL)\frac{dT}{dr}$ (Fourier equation)

Rearranging and integrating between the limits $T=T_1$ at $r=r_1$ and $T=T_2$ at $r=r_2$ as,

$$Q\int_{r_1}^{r_2}\frac{dr}{r} = -2\pi kL\int_{T_1}^{T_2} dT$$

$$Q[\ln r]_{r_1}^{r_2} = -2\pi kL[T]_{T_1}^{T_2}$$

$$\therefore Q = \frac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)} = \frac{T_1 - T_2}{\frac{\ln(r_2/r_1)}{2\pi kL}} = \frac{\Delta T}{R}$$

Question 18. A long hollow cylinder with thermal conductivity $80\, W/mK$ has inner and outer radii as 10 cm and 20 cm, respectively. If its inner and outer surfaces are maintained at constant temperatures of 420 K and 320 K, respectively, then calculate (i) the steady rate of heat loss through the pipe per unit of its length and (ii) the temperature at a point halfway between the inner and the outer surfaces.

Solution: Let $k = 80\, W/mK$, $r_1 = 10\, cm = 0.1\, m$, $r_2 = 20\, cm = 0.2\, m$, $T_1 = 420\, K$, $T_2 = 320\, K$, $L = 1\, m$ and r be the radius at halfway between the inner and outer surfaces.

(i) $$Q = \frac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)} = \frac{2\pi \times 80 \times 1 \times (420 - 320)}{\ln(0.2/0.1)} = 72517.76\, W$$ **(Ans.)**

(ii) $$r = \frac{r_1 + r_2}{2} = \frac{0.1 + 0.2}{2} = 0.15\, m$$

Since $$T = \frac{T_2 - T_1}{\ln(r_2/r_1)}\ln r + T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)}\ln r_1$$

$$\therefore T = \frac{320 - 420}{\ln(0.2/0.1)}\ln 0.15 + 420 - \frac{320 - 420}{\ln(0.2/0.1)}\ln 0.1 = 361.5\, K$$ **(Ans.)**

Question 19. A long hollow cylindrical tube has inner and outer radii as 4 cm and 8 cm, respectively. A current of 5 ampere flows in the electric heater fitted into it along its length to maintain a steady temperature difference of 373 K between the inner and outer surfaces. If resistance of the heater is 20 ohm per meter of its length then find the thermal conductivity of tube material.

Solution: Let $r_1 = 4\, cm = 0.04\, m$, $r_2 = 8\, cm = 0.08\, m$, $I = 5\, A$, $\Delta T = 373\, K$, and $R_e = 20\, \Omega/m$.

Since $$\frac{Q}{L} = \frac{2\pi k\Delta T}{\ln(r_2/r_1)} = I^2 \times \frac{R_e}{L}$$

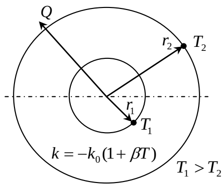
$$\therefore k = \frac{I^2 (R_e/L) \times \ln(r_2/r_1)}{2\pi\Delta T} = \frac{5^2 \times 20 \times \ln(0.08/0.04)}{2\pi \times 373} = 0.148\, W/mK$$ **(Ans.)**

Question 20. Derive expression for rate of heat conduction through a hollow cylinder with variable thermal conductivity.

Answer: Rate of heat conduction through a hollow cylinder with variable thermal conductivity can be obtained by using $k = -k_0(1 + \beta T)$ in Fourier Law as,

$$Q = -k_0(1+\beta T) \times 2\pi r L \times \frac{dT}{dr}$$

Rearranging and integrating both sides within the boundary conditions (i) $T = T_1$ at $r = r_1$ and (ii) $T = T_2$ at $r = r_2$ as shown in below Figure.



$$Q\int_{r_1}^{r_2} \frac{dr}{r} = -2\pi L k_0 \int_{T_1}^{T_2} (1+\beta T)dT \Rightarrow Q[\ln r]_{r_1}^{r_2} = -2\pi L k_0 \left[T + \beta \frac{T^2}{2}\right]_{T_1}^{T_2}$$

$$Q = \frac{2\pi L k_0}{\ln(r_2/r_1)}\left[\left(T_1 + \beta\frac{T_1^2}{2}\right) - \left(T_2 + \beta\frac{T_2^2}{2}\right)\right] \Rightarrow Q = \frac{2\pi L k_0}{\ln(r_2/r_1)}\left[(T_1 - T_2) + \beta\frac{(T_1^2 - T_2^2)}{2}\right]$$

$$Q = \frac{2\pi L k_0}{\ln(r_2/r_1)}(T_1 - T_2)\left[1 + \beta\frac{(T_1 + T_2)}{2}\right] \Rightarrow Q = \frac{2\pi L \times k_0(1+\beta T_m) \times (T_1 - T_2)}{\ln(r_2/r_1)}$$

The mean thermal conductivity $k_m = k_0(1+\beta T_m)$ is evaluated at mean temperature $T_m = (T_1 + T_2)/2$.

$$\therefore Q = \frac{2\pi L k_m (T_1 - T_2)}{\ln(r_2/r_1)}$$

Question 21. The inside and outside radii of 4 m long thick walled copper pipe are 2 cm and 4 cm respectively. If the inner and outer surfaces are maintained at 580 K and 550 K, respectively and thermal conductivity of pipe varies with temperature T in K as per the linear relation, $k = [380 \times \{1 - 0.00008(T - 170)\}]\, W/mK$, determine the steady rate of heat loss through the pipe.

Solution: Let $L = 4\,m$, $r_1 = 2\,cm = 0.02\,m$, $r_2 = 4\,cm = 0.04\,m$, $T_1 = 580\,K$, $T_2 = 550\,K$ and $k = [380 \times \{1 - 0.00008(T - 170)\}]\, W/mK$.

Since $k_m = k_0(1+\beta T_m)$

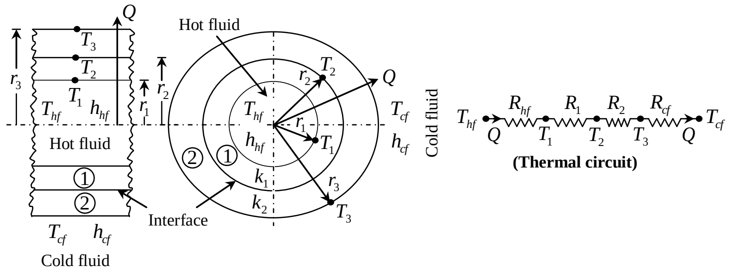
Here $T_m = \dfrac{(T_1 - 170) + (T_2 - 170)}{2} = \dfrac{(580-170) + (550-170)}{2} = 395\,K$

$$\therefore k_m = 380 \times [1 - 0.00008 T_m] = 380 \times [1 - 0.00008 \times 395] = 367.99\, W/mK$$

$$Q = \frac{2\pi L k_m (T_1 - T_2)}{\ln(r_2/r_1)} = \frac{2\pi \times 4 \times 367.99 \times (580 - 550)}{\ln(0.04/0.02)} = 400.29\, kW$$ **(Ans.)**

Question 22. For one dimensional steady state heat conduction derive expressions for rate of heat conduction through a composite cylinder without internal heat generation. Also find the expressions for overall heat transfer coefficient.

Answer: Consider a composite cylinder or coaxial cylinders (e.g., lagged pipe carrying steam) of length L consisting of two layers 1 and 2 having a perfect contact at the interfaces as shown in below Figure. Hot fluid is flowing inside while the outer surface is exposed to the cold fluid. Let steady radial heat conduction (one-dimensional) occurs through the composite cylinder without any heat generation.



Let r_1 be the inner radius, r_2, r_3 be outer radii of the two layers 1 and 2, respectively, and k_1, k_2 be the constant thermal conductivities of the corresponding layers,

$A_i = 2\pi r_1 L$ and $A_o = 2\pi r_3 L$ be the internal and external surface areas of the composite cylinder, respectively,

T_{hf}, T_{cf} be the temperatures and h_{hf}, h_{cf} be the convective heat transfer coefficients of hot and cold fluids, respectively,

T_1 and T_3 be the surface temperatures of the composite cylinder on hot fluid side and cold fluid side, respectively, and T_2 be the temperature at the interface.

Under steady state conditions, the rate of heat transfer from the hot to cold fluid through each layer remains constant and it can be given as,

$$Q = h_{hf} A_i (T_{hf} - T_1) = \frac{2\pi k_1 L(T_1 - T_2)}{\ln(r_2/r_1)} = \frac{2\pi k_2 L(T_2 - T_3)}{\ln(r_3/r_2)} = h_{hf} A_o (T_3 - T_{cf})$$

or $$Q = \frac{T_{hf} - T_1}{\frac{1}{A_i h_{hf}}} = \frac{T_1 - T_2}{\frac{\ln(r_2/r_1)}{2\pi k_1 L}} = \frac{T_2 - T_3}{\frac{\ln(r_3/r_2)}{2\pi k_2 L}} = \frac{T_3 - T_{cf}}{\frac{1}{A_o h_{cf}}}$$

Here, various thermal resistances are given as,

$$R_{hf} = \frac{1}{A_i h_{hf}};\ R_1 = \frac{\ln(r_2/r_1)}{2\pi k_1 L};\ R_2 = \frac{\ln(r_3/r_2)}{2\pi k_2 L};\ R_{cf} = \frac{1}{A_o h_{cf}}$$

Thus $$Q = \frac{T_{hf} - T_1}{R_{hf}} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_3}{R_2} = \frac{T_3 - T_{cf}}{R_{cf}}$$

From above Equation, we get:

$$T_{hf} - T_1 = QR_{hf}\,;\; T_1 - T_2 = QR_1\,;\; T_2 - T_3 = QR_2\,;\; T_3 - T_{cf} = QR_{cf}$$

Adding, we get: $T_{hf} - T_{cf} = Q[R_{hf} + R_1 + R_2 + R_{cf}]$

$$\therefore Q = \frac{T_{hf} - T_{cf}}{R_{hf} + R_1 + R_2 + R_{cf}} = \frac{\Delta T}{\sum R} \qquad \text{(i)}$$

The rate of heat transfer from the hot to cold fluid can also be written as,

$$Q = UA(T_{hf} - T_{cf}) = UA\Delta T \qquad \text{(ii)}$$

Here, U is the overall heat transfer coefficient and A is the area normal to the direction of heat flow.

By comparing Equations (i) and (ii), we obtain:

$$U = \frac{\Delta T}{A\sum R}$$

For a hollow cylinder the area varies with radius. Therefore, it becomes necessary to specify the area on which U is based. Thus depending on whether the inner or outer area is specified, two values are defined for U are the overall heat transfer coefficient based on inner area (U_i) and the overall heat transfer coefficient based on outer area (U_o) as,

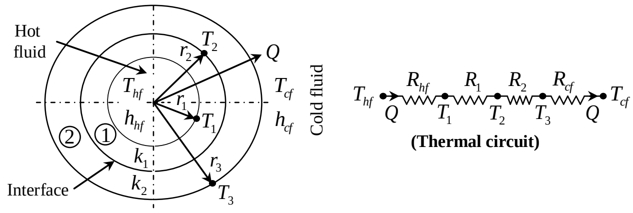
$$U_i = \frac{1}{A_i\sum R} = \frac{1}{\left[\dfrac{1}{h_{hf}} + \dfrac{r_1}{k_1}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{r_1}{k_2}\ln\left(\dfrac{r_3}{r_2}\right) + \dfrac{r_1}{r_3}\dfrac{1}{h_{cf}}\right]}$$

and $$U_o = \frac{1}{A_o\sum R} = \frac{1}{\left[\dfrac{r_3}{r_1}\dfrac{1}{h_{hf}} + \dfrac{r_3}{k_1}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{r_3}{k_2}\ln\left(\dfrac{r_3}{r_2}\right) + \dfrac{1}{h_{cf}}\right]}$$

The concept of overall heat transfer coefficient in cylindrical system is used in the design of heat exchangers.

Question 23. A steel pipe $(k = 55\,W/mK)$ of inner diameter 10 cm and 11 cm outer diameter is insulated with a material $(k = 0.15\,W/mK)$ of thickness 3.5 cm. The pipe carries a hot fluid at a temperature of 360 K with a convective heat transfer coefficient of $600\,W/m^2K$. If outside temperature and convective heat transfer coefficient are 298 K and $10\,W/m^2K$, respectively, calculate (i) the steady loss of heat per meter length of pipe and (ii) the overall heat transfer coefficient based on inner surface.

Answer: Refer below Figure. Let $k_1 = 55\,W/mK$, $d_1 = 10\,cm = 0.1\,m$, $d_2 = 11\,cm = 0.11\,m$, $k_2 = 0.15\,W/mK$, $t = 3.5\,cm = 0.035\,m$, $T_{hf} = 360\,K$, $h_{hf} = 600\,W/m^2K$, $T_{cf} = 298\,K$ and $h_{cf} = 10\,W/m^2K$.



$r_1 = d_1/2 = 0.1/2 = 0.05\,m$; $r_2 = d_2/2 = 0.11/2 = 0.055\,m$

$r_3 = r_2 + t = 0.055 + 0.035 = 0.09\,m$

(i) The steady loss of heat per meter length of pipe can be given as below.

$$\frac{Q}{L} = \frac{2\pi(T_{hf} - T_{cf})}{\left[\frac{1}{h_{hf} r_1} + \frac{\ln(r_2/r_1)}{k_1} + \frac{\ln(r_3/r_2)}{k_2} + \frac{1}{h_{cf} r_3}\right]}$$

Thus $$\frac{Q}{L} = \frac{2\pi \times (360 - 298)}{\left[\frac{1}{600 \times 0.05} + \frac{\ln(0.055/0.05)}{55} + \frac{\ln(0.09/0.055)}{0.15} + \frac{1}{10 \times 0.09}\right]}$$

$$\therefore \frac{Q}{L} = 87.95\,W/m$$ **(Ans.)**

(ii) The overall heat transfer coefficient based on inner surface can be given as below.

$$U_i = \frac{1}{\left[\frac{1}{h_{hf}} + \frac{r_1}{k_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_2}\ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{r_3}\frac{1}{h_{cf}}\right]}$$

Thus $$U_i = \frac{1}{\left[\frac{1}{600} + \frac{0.05}{55}\ln\left(\frac{0.055}{0.05}\right) + \frac{0.05}{0.15}\ln\left(\frac{0.09}{0.055}\right) + \frac{0.05}{0.09} \times \frac{1}{10}\right]}$$

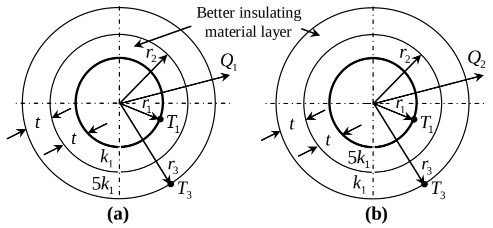
$$\therefore U_i = 4.515\,W/m^2K$$ **(Ans.)**

Question 24. A 3 cm outer diameter steam pipe is to be covered with two layers of insulation each having thickness 2.5 cm. The average thermal conductivity of one material is five times of the other. Determine the percentage decrease in heat transfer, if better insulating material is kept next to pipe surface than it is as outer layer. Assume that the outside and inside temperatures are fixed and other conditions remain same.

Solution: Refer below Figure. Let $d_1 = 3\,cm = 0.03\,m$, $t_1 = t_2 = t = 2.5\,cm = 0.025\,m$, k_1 be the thermal conductivity of first layer (better insulating material), $5k_1$ be the thermal conductivity of second layer, T_1 be the inside temperature and T_3 be the outside temperature.

$r_1 = d_1/2 = 0.03/2 = 0.015\,m$; $r_2 = r_1 + t = 0.015 + 0.025 = 0.04\,m$;

$r_3 = r_2 + t = 0.04 + 0.025 = 0.065\,m$



Case I: When better insulating material layer is kept inside. Refer Figure (a).

$$Q_1 = \frac{2\pi L(T_1 - T_3)}{\frac{\ln(r_2/r_1)}{k_1} + \frac{\ln(r_3/r_2)}{5k_1}} = \frac{2\pi L\Delta T}{\frac{\ln(0.04/0.015)}{k_1} + \frac{\ln(0.065/0.04)}{5k_1}} = 1.855\pi k_1 L\Delta T$$

Case II: When better insulating material layer is kept outside. Refer Figure (b).

$$Q_2 = \frac{2\pi L(T_1 - T_3)}{\frac{\ln(r_2/r_1)}{5k_1} + \frac{\ln(r_3/r_2)}{k_1}} = \frac{2\pi L\Delta T}{\frac{\ln(0.04/0.015)}{5k_1} + \frac{\ln(0.065/0.04)}{k_1}} = 2.934\pi k_1 L\Delta T$$

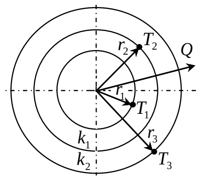
Since $Q_2 > Q_1$, therefore, better insulating material layer next to the pipe decreases the heat transfer rate. The percentage decrease in heat transfer rate is obtained as below.

$$\frac{Q_2 - Q_1}{Q_2}\times 100 = \frac{2.934 - 1.855}{2.934}\times 100 = 36.8\%$$ **(Ans.)**

Comment: For the insulation of a cylindrical pipe, material with lower thermal conductivity should be applied next to the hot surface.

Question 25. A steel pipe $(k = 30\,W/mK)$ of inner diameter 5 cm and 6 cm outer diameter is insulated with a material of thermal conductivity $0.056\,W/mK$. If the temperature at the interface between pipe and insulation is 585 K, while the temperature on outside surface of insulation must not exceed 345 K, with permissible heat loss of $705\,W/m$ then, calculate (i) the required thickness of insulation and (ii) the temperature at the inside surface of pipe.

Solution: Refer below Figure. Let $k_1 = 30\,W/mK$, $d_1 = 5\,cm = 0.05\,m$, $d_2 = 6\,cm = 0.06\,m$, $k_2 = 0.056\,W/mK$, $T_2 = 585\,K$, $T_3 = 345\,K$ and $(Q/L) = 705\,W/m$. Let t be the thickness of insulation and T_1 be the temperature at the inside surface of pipe.



(i) $r_1 = d_1/2 = 0.05/2 = 0.025\,m$; $r_2 = d_2/2 = 0.06/2 = 0.03\,m$

The steady rate of heat transfer through insulation layer can be given as below.

$$\frac{Q}{L} = \frac{2\pi k_2(T_2 - T_3)}{\ln(r_3/r_2)} \text{ or } 705 = \frac{2\pi \times 0.056 \times (585 - 345)}{\ln(r_3/0.03)}$$

$$\ln\left(\frac{r_3}{0.03}\right) = \frac{2\pi \times 0.056 \times (585-345)}{705} = 0.1198$$

$$\therefore r_3 = e^{0.1198} \times 0.03 = 0.0338\,m$$

$$\therefore t = r_3 - r_2 = 0.0338 - 0.03 = 0.0038\,m \;or\; 3.8\,mm$$ **(Ans.)**

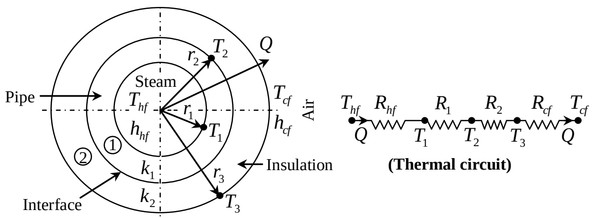
(ii) The steady rate of heat transfer through the pipe can be given as below.

$$\frac{Q}{L} = \frac{2\pi k_1 (T_1 - T_2)}{\ln(r_2 / r_1)} \quad \text{or} \quad 705 = \frac{2\pi \times 30 \times (T_1 - 585)}{\ln(0.03/0.025)}$$

$$\therefore T_1 = \frac{705 \times \ln(0.03/0.025)}{2\pi \times 30} + 585 = 585.68\,K$$ **(Ans.)**

Question 26. A 5 m long steam pipe $(k = 45\,W/mK)$ of 4.8 cm inside diameter and 6.4 cm outside diameter is insulated with a 2.6 cm radial thickness of high temperature insulation $(k = 1.1\,W/mK)$. The surface heat transfer coefficient for inside and outside surfaces are $4650\,W/m^2K$ and $11.5\,W/m^2K$, respectively. If the steam temperature is 473 K and ambient temperature is 298 K, determine (i) rate of heat loss from pipe, (ii) temperature at the interface and (iii) overall heat transfer coefficients.

Solution: Refer below Figure. Let $L = 5\,m$, $k_1 = 45\,W/mK$, $d_1 = 4.8\,cm = 0.048\,m$, $d_2 = 6.4\,cm = 0.064\,m$, $t = 2.6\,cm = 0.026\,m$, $k_2 = 1.1\,W/mK$, $h_{hf} = 4650\,W/m^2K$, $h_{cf} = 11.5\,W/m^2K$, $T_{hf} = 473\,K$ and $T_{cf} = 298\,K$.



$$r_1 = d_1/2 = 0.048/2 = 0.024\,m;\; r_2 = d_2/2 = 0.064/2 = 0.032\,m$$

$$r_3 = r_2 + t = 0.032 + 0.026 = 0.058\,m$$

$$R_{hf} = \frac{1}{h_{hf} A_i} = \frac{1}{2\pi r_1 h_{hf} L} = \frac{1}{2\pi \times 0.024 \times 4650 \times 5} = 2.852 \times 10^{-4}\; K/W$$

$$R_1 = \frac{\ln(r_2 / r_1)}{2\pi k_1 L} = \frac{\ln(0.032/0.024)}{2\pi \times 45 \times 5} = 2.035 \times 10^{-4}\; K/W$$

$$R_2 = \frac{\ln(r_3 / r_2)}{2\pi k_2 L} = \frac{\ln(0.058/0.032)}{2\pi \times 1.1 \times 5} = 0.01721\,K/W$$

$$R_{cf} = \frac{1}{h_{cf} A_o} = \frac{1}{2\pi r_3 h_{cf} L} = \frac{1}{2\pi \times 0.058 \times 11.5 \times 5} = 0.04772\,K/W$$

Thus $\sum R = 2.852\times10^{-4} + 2.035\times10^{-4} + 0.01721 + 0.04772 = 0.06542\,K/W$

(i) The rate of heat loss from pipe can be evaluated as follows.

$$Q = \frac{T_{hf} - T_{cf}}{\sum R} = \frac{473-298}{0.06542} = 2675.02\,W$$ **(Ans.)**

(ii) The steady rate of heat transfer through the pipe can be given as below.

$$Q = \frac{T_{hf} - T_2}{R_{hf} + R_1}$$ (Considering first two thermal resistances)

Thus $T_2 = T_{hf} - Q\times(R_{hf} + R_1)$

$$\therefore T_2 = 473 - 2675.02\times(2.852\times10^{-4} + 2.035\times10^{-4}) = 471.69\,K$$ **(Ans.)**

(iii) The overall heat transfer coefficient based on inner area can be evaluated as below.

$$U_i = \frac{1}{A_i\sum R} = \frac{1}{2\pi r_1 L\sum R} = \frac{1}{2\pi\times0.024\times5\times0.06542} = 20.27\,W/m^2K$$ **(Ans.)**

The overall heat transfer coefficient based on outer area can be evaluated as below.

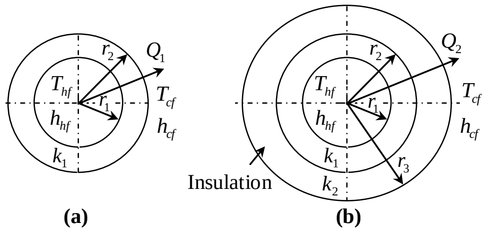
$$U_o = \frac{1}{A_o\sum R} = \frac{1}{2\pi r_3 L\sum R} = \frac{1}{2\pi\times0.058\times5\times0.06542} = 8.39\,W/m^2K$$ **(Ans.)**

Note: $U_i > U_o$ as $A_i < A_o$

Question 27. Saturated steam at 385 K flows inside a copper pipe $(k = 401\,W/mK)$ of length 3 m having inner diameter of 8 cm and outer diameter of 10 cm. The surface heat transfer coefficient for inside and outside surfaces are $10000\,W/m^2K$ and $15\,W/m^2K$, respectively. Determine the steady rate of heat loss from pipe if it is located in a room at 300 K. How this heat loss rate would be affected if pipe is lagged with 5 cm thick insulation $(k = 0.2\,W/mK)$.

Solution: Refer below Figure. Let $T_{hf} = 385\,K$, $k_1 = 401\,W/mK$, $L = 3\,m$, $d_1 = 8\,cm = 0.08\,m$, $d_2 = 10\,cm = 0.1\,m$, $h_{hf} = 10000\,W/m^2K$, $h_{cf} = 15\,W/m^2K$, $T_{cf} = 300\,K$, $t = 5\,cm = 0.05\,m$ and $k_2 = 0.2\,W/mK$.

$r_1 = d_1/2 = 0.08/2 = 0.04\,m$; $r_2 = d_2/2 = 0.1/2 = 0.05\,m$; $r_3 = r_2 + t = 0.05 + 0.05 = 0.1\,m$



Case I: When the pipe is not insulated. Refer Figure (a)

$$Q_1 = \frac{2\pi L(T_{hf} - T_{cf})}{\frac{1}{r_1 h_{hf}} + \frac{\ln(r_2/r_1)}{k_1} + \frac{1}{r_2 h_{cf}}}$$

$$\therefore Q_1 = \frac{2\pi \times 3 \times (385 - 300)}{\frac{1}{0.04 \times 10000} + \frac{\ln(0.05/0.04)}{401} + \frac{1}{0.05 \times 15}} = 1198.91 W$$

Case II: When the pipe is insulated. Refer Figure (b)

$$Q_2 = \frac{2\pi L(T_{hf} - T_{cf})}{\frac{1}{r_1 h_{hf}} + \frac{\ln(r_2/r_1)}{k_1} + \frac{\ln(r_3/r_2)}{k_2} + \frac{1}{r_3 h_{cf}}}$$

$$\therefore Q_2 = \frac{2\pi \times 3 \times (385 - 300)}{\frac{1}{0.04 \times 10000} + \frac{\ln(0.05/0.04)}{401} + \frac{\ln(0.1/0.05)}{0.2} + \frac{1}{0.05 \times 15}} = 333.65 W$$

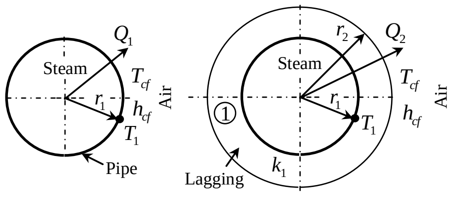
Since $Q_2 < Q_1$, therefore, addition of insulation decreases the heat transfer rate. The percentage decrease in heat transfer rate is obtained as below.

$$\frac{Q_1 - Q_2}{Q_1} \times 100 = \frac{1198.91 - 333.65}{1198.91} \times 100 = 72.17\%$$ **(Ans.)**

Question 28. A steel pipe having an external diameter of 40 mm carrying steam has its outer surface at 475 K. It is to be lagged with a layer of insulating material of thermal conductivity $0.08 W/mK$ to reduce the heat loss by 60%. The ambient temperature is 304 K and the convective heat transfer coefficient is $10 W/m^2K$. What thickness of lagging must be added when all other conditions remain unchanged? Neglect resistance due to pipe material.

Solution: Refer below Figure. Let $d_1 = 40\,mm = 0.04\,m$, $T_1 = 475\,K$, $k_1 = 0.08\,W/mK$, $Q_2 = 0.4\,Q_1$, $T_{cf} = 304\,K$, $h_{cf} = 10\,W/m^2K$, $L = 1\,m$, r_2 be the radius of pipe after adding the insulation and $t = (r_2 - r_1)$ be the thickness of lagging.

$$r_1 = d_1/2 = 0.04/2 = 0.02\,m$$



The rate of heat loss from pipe without any insulation as shown in Figure (a) can be given as below.

$$Q_1 = 2\pi r_1 h_{cf} L(T_1 - T_{cf}) = 2\pi \times 0.02 \times 10 \times 1 \times (475 - 304) = 214.9\,W$$

Since addition of lagging (Figure (b)) reduces heat loss by 60%, therefore, allowable heat loss will become 40% of Q_1, i.e., $Q_2 = 0.4\,Q_1 = 0.4 \times 214.9 = 85.96\,W$.

Since $Q_2 = \dfrac{2\pi L(T_1 - T_{cf})}{\dfrac{\ln(r_2/r_1)}{k_1} + \dfrac{1}{r_2 h_{cf}}}$ Thus $85.96 = \dfrac{2\pi \times 1 \times (475-304)}{\dfrac{\ln(r_2/0.02)}{0.08} + \dfrac{1}{r_2 \times 10}}$

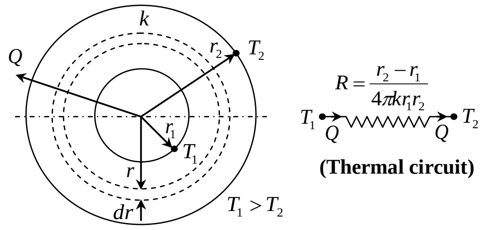
or $12.5 \ln\left(\dfrac{r_2}{0.02}\right) + \dfrac{0.1}{r_2} = 12.49$

This equation is to be solved for r_2 by trial and error, which would give $r_2 = 0.0456 m$.

$$\therefore t = r_2 - r_1 = 0.0456 - 0.02 = 0.0256 m \; or \; 25.6 \; mm$$ **(Ans.)**

Question 29. For one dimensional steady state heat conduction derive expressions for temperature distribution and rate of heat conduction through a hollow sphere with constant thermal conductivity and without internal heat generation.

Answer: Consider steady state one dimensional (radial direction) heat conduction through a hollow sphere. Let r_1 be the inner radius, r_2 be the outer radius, T_1 and T_2 be the constant temperatures at inner and outer surfaces, respectively, such that $T_1 > T_2$, k be the constant thermal conductivity and there is no heat generation (Refer below Figure). Here, heat flows only in radial direction, therefore, temperature distribution is $T(r)$.



The general heat conduction equation in spherical coordinates is given by,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right] + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Since (i) $\partial^2 T/\partial \phi^2 = 0$ and $\partial^2 T/\partial z^2 = 0$ (1-D heat transfer in radial direction only), (ii) $\partial T/\partial t = 0$ (Steady heat transfer) and (iii) $q_g = 0$ (No heat generation), therefore, we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad \text{or} \quad \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad [\because (1/r^2) \neq 0]$$

Integrating, we get: $r^2 \dfrac{dT}{dr} = C_1$ or $dT = C_1 \dfrac{dr}{r^2}$

Again integrating, we get: $T = -\dfrac{C_1}{r} + C_2$

The values of arbitrary constants C_1 and C_2 can be obtained with the help of the following two boundary conditions.

(i) $T = T_1$ at $r = r_1$ and (ii) $T = T_2$ at $r = r_2$

Applying first boundary conditions, we get: $T_1 = -\frac{C_1}{r_1} + C_2$ (i)

Applying second boundary conditions, we get: $T_2 = -\frac{C_1}{r_2} + C_2$ (ii)

From the expressions (i) and (ii), we obtain the values of C_1 and C_2 as below.

$$C_1 = \frac{T_1 - T_2}{(1/r_2) - (1/r_1)} \text{ and } C_2 = T_1 + \frac{1}{r_1}.\frac{T_1 - T_2}{(1/r_2) - (1/r_1)}$$

The expression for temperature distribution becomes,

$$T = -\frac{1}{r}.\frac{T_1 - T_2}{(1/r_2) - (1/r_1)} + T_1 + \frac{1}{r_1}.\frac{T_1 - T_2}{(1/r_2) - (1/r_1)}$$

Differentiating above Equation with respect to r, we obtain the below expression.

$$\frac{dT}{dr} = \frac{r_1 r_2 (T_2 - T_1)}{r^2 (r_2 - r_1)}$$

The rate of heat conduction through the sphere is given by Fourier's law as,

$$Q = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{r_1 r_2 (T_2 - T_1)}{r^2 (r_2 - r_1)} = 4\pi k r_1 r_2 \frac{T_1 - T_2}{(r_2 - r_1)} = \frac{T_1 - T_2}{\frac{(r_2 - r_1)}{4\pi k r_1 r_2}} = \frac{\Delta T}{R}$$

The term $\frac{(r_2 - r_1)}{4\pi k r_1 r_2}$ in above Equation is called the conductive thermal resistance or simply thermal resistance $(R_{cond}\ or\ R)$ for the hollow sphere. The thermal circuit is also shown in above Figure.

Alternatively:

$$Q = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr} \quad \text{(Fourier equation)}$$

Separating the Fourier equation and integrating between the limits $T = T_1$ at $r = r_1$ and $T = T_2$ at $r = r_2$ as below.

$$Q\int_{r_1}^{r_2} \frac{dr}{r^2} = -4\pi k \int_{T_1}^{T_2} dT \Rightarrow -Q\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = -4\pi k (T_2 - T_1)$$

$$\therefore Q = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)} = \frac{T_1 - T_2}{\frac{(r_2 - r_1)}{4\pi k r_1 r_2}} = \frac{\Delta T}{R}$$

Question 30. A hollow spherical container $(k = 55 W/mK)$ has inner and outer radii as 6 cm and 14 cm, respectively. If its inner and outer surfaces are maintained at constant temperatures of 550 K and 350 K, respectively, then calculate (i) the steady rate of heat loss through this container and (ii) the temperature at a point halfway between the inner and the outer surfaces.

Solution: Let $k = 55\,W/mK$, $r_1 = 6\,cm = 0.06\,m$, $r_2 = 14\,cm = 0.14\,m$, $T_1 = 550\,K$ and $T_2 = 350\,K$.

(i) $$Q = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)} = \frac{4\pi \times 55 \times 0.06 \times 0.14 \times (550-350)}{(0.14-0.06)} = 14.514\,kW$$ **(Ans.)**

$$r = \frac{r_1 + r_2}{2} = \frac{0.06+0.14}{2} = 0.1\,m$$

(ii) $$T = T_1 + \frac{r_2}{r}\left(\frac{r - r_1}{r_2 - r_1}\right)(T_2 - T_1)$$

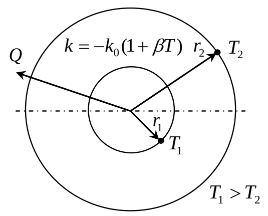
$$\therefore T = 550 + \frac{0.14}{0.1} \times \left(\frac{0.1-0.06}{0.14-0.06}\right) \times (350-550) = 410\,K$$ **(Ans.)**

Question 31. Derive expression for rate of heat conduction through a hollow sphere with variable thermal conductivity.

Answer: The rate of heat conduction through a hollow cylinder with variable thermal conductivity can be obtained by substituting the value of $k = -k_0(1+\beta T)$ in Fourier Law as,

$$Q = -kA\frac{dT}{dr} = -k_0(1+\beta T) \times 4\pi r^2 \times \frac{dT}{dr}$$

Rearranging the above expression and integrating both sides within the boundary conditions (i) $T = T_1$ at $r = r_1$ and (ii) $T = T_2$ at $r = r_2$ as shown in below Figure.



$$Q\int_{r_1}^{r_2} \frac{dr}{r^2} = -4\pi k_0 \int_{T_1}^{T_2} (1+\beta T)dT \quad \text{or} \quad Q\left[-\frac{1}{r}\right]_{r_1}^{r_2} = -4\pi k_0 \left[T + \beta\frac{T^2}{2}\right]_{T_1}^{T_2}$$

$$Q = \frac{4\pi k_0}{(1/r_1) - (1/r_2)}\left[\left(T_1 + \beta\frac{T_1^2}{2}\right) - \left(T_2 + \beta\frac{T_2^2}{2}\right)\right]$$

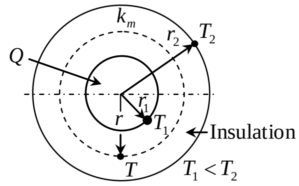
$$Q = \frac{4\pi r_1 r_2 k_0}{(r_2 - r_1)}(T_1 - T_2)\left[1 + \beta\frac{(T_1 + T_2)}{2}\right] \Rightarrow Q = \frac{4\pi r_1 r_2 \times k_0(1+\beta T_m) \times (T_1 - T_2)}{(r_2 - r_1)}$$

The mean thermal conductivity $k_m = k_0(1+\beta T_m)$ is evaluated at mean temperature $T_m = (T_1 + T_2)/2$.

$$\therefore Q = \frac{4\pi k_m r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)} = \frac{(T_1 - T_2)}{\dfrac{(r_2 - r_1)}{4\pi k_m r_1 r_2}} = \frac{\Delta T}{R}$$

Question 32. A thin walled spherical container of 140 mm inner radius is insulated by 100 mm thick insulation whose thermal conductivity varies with temperature T in K as per the linear relation, $k(W/mK) = [0.03 \times (1 + 0.006T)]$. If the inner and outer surfaces are maintained at -186 °C and 20 °C, respectively, determine (i) the steady rate of heat inflow and (ii) the temperature at mean radius.

Solution: Refer below Figure. Let $r_1 = 140\,mm = 0.14\,m$, $t = 100\,mm = 0.1\,m$, $k = [0.03(1 + 0.006T)]\,W/mK$, $T_1 = -186\,^oC = 87\,K$ and $T_2 = 20\,^oC = 293\,K$.



(i) $$T_m = \frac{T_1 + T_2}{2} = \frac{87 + 293}{2} = 190\,K$$

$$k_m = k_0(1 + \beta T_m) = 0.03 \times [1 + 0.006 \times 190] = 0.0642\,W/mK$$

$$r_2 = r_1 + t = 0.14 + 0.1 = 0.24\,m$$

$$\Delta T = T_2 - T_1 = 293 - 87 = 206\,K$$

The steady rate of heat inflow can be evaluated as follows.

$$Q = \frac{4\pi k_m r_1 r_2 \Delta T}{r_2 - r_1} = \frac{4\pi \times 0.0642 \times 0.14 \times 0.24 \times 206}{0.24 - 0.14} = 55.84\,W$$ **(Ans.)**

(ii) Let T be the temperature at the mean radius, r.

$$r = \frac{r_1 + r_2}{2} = \frac{0.14 + 0.24}{2} = 0.19\,m$$

$$T_m = \frac{T_1 + T}{2} = \frac{87 + T}{2}$$

$$\therefore k_m = 0.03 \times \left(1 + 0.006 \times \frac{87 + T}{2}\right) = (0.03783 + 0.00009T)\,W/mK$$

$$\Delta T = T - T_1 = (T - 87)\,K$$

Since $$Q = \frac{4\pi k_m r_1 r \Delta T}{r - r_1}$$

Thus $$55.84 = \frac{4\pi \times (0.03783 + 0.00009T) \times 0.14 \times 0.19 \times (T - 87)}{0.19 - 0.14}$$

$$8.353 = 0.000009T^2 + 0.03783T - 3.291 - 0.00783T$$

Thus $\quad 0.000009T^2 + 0.03T - 11.644 = 0$

$$\therefore T = \frac{-0.03 \pm \sqrt{0.03^2 + 4 \times 0.00009 \times 11.644}}{2 \times 0.00009} = 229.8\ K$$ (Taking +ve value)

Therefore, the temperature at mean-radius is $(273 - 229.8) = -43.2\ ^oC$ **(Ans.)**

Question 33. For one dimensional steady state heat conduction derive expressions for rate of heat conduction through a composite sphere without internal heat generation. Also find the expressions for overall heat transfer coefficient.

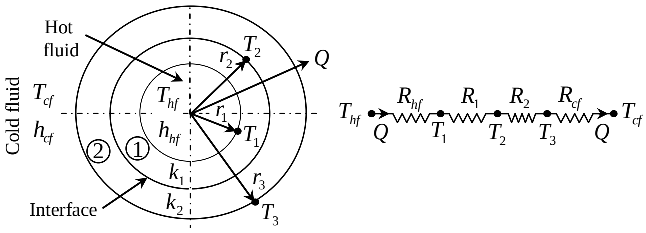
Answer: Consider a composite sphere or coaxial spheres consisting of two layers 1 and 2 having a perfect contact at the interface as shown in below Figure. Hot fluid is flowing inside while the outer surface is exposed to the cold fluid. Let steady radial heat conduction (one-dimensional) occurs through the composite sphere without any heat generation.

Let r_1 be the inner radius, r_2, r_3 be outer radii of the two layers 1 and 2, respectively, and k_1, k_2 be the constant thermal conductivities of the corresponding layers,

$A_i = 4\pi r_1^2$ and $A_o = 4\pi r_3^2$ be the internal and external surface areas of the composite sphere, respectively,

T_{hf}, T_{cf} be the temperatures and h_{hf}, h_{cf} be the convective heat transfer coefficients of hot and cold fluids, respectively,

T_1 and T_3 be the surface temperatures of the composite sphere on hot fluid and cold fluid sides, respectively, and T_2 be the temperature at the interface.



Under steady state conditions, the rate of heat transfer from the hot to cold fluid through each layer remains constant and it can be given by the following expressions.

$$Q = h_{hf} A_i (T_{hf} - T_1) = \frac{4\pi k_1 r_1 r_2 (T_1 - T_2)}{r_2 - r_1} = \frac{4\pi k_2 r_2 r_3 (T_2 - T_3)}{r_3 - r_2} = h_{cf} A_o (T_3 - T_{cf})$$

or $$Q = \frac{T_{hf} - T_1}{\dfrac{1}{A_i h_{hf}}} = \frac{T_1 - T_2}{\dfrac{r_2 - r_1}{4\pi k_1 r_1 r_2}} = \frac{T_2 - T_3}{\dfrac{r_3 - r_2}{4\pi k_2 r_2 r_3}} = \frac{T_3 - T_{cf}}{\dfrac{1}{A_o h_{cf}}}$$

or $$Q=\frac{T_{hf}-T_1}{\frac{1}{4\pi r_1^{2}h_{hf}}}=\frac{T_1-T_2}{\frac{r_2-r_1}{4\pi k_1r_1r_2}}=\frac{T_2-T_3}{\frac{r_3-r_2}{4\pi k_2r_2r_3}}=\frac{T_3-T_{cf}}{\frac{1}{4\pi r_3^{2}h_{cf}}}$$

Here, various thermal resistances are given as below.

$$R_{hf}=\frac{1}{4\pi r_1^{2}h_{hf}};\ R_1=\frac{r_2-r_1}{4\pi k_1r_1r_2};\ R_2=\frac{r_3-r_2}{4\pi k_2r_2r_3};\ R_{cf}=\frac{1}{4\pi r_3^{2}h_{cf}}$$

Thus $$Q=\frac{T_{hf}-T_1}{R_{hf}}=\frac{T_1-T_2}{R_1}=\frac{T_2-T_3}{R_2}=\frac{T_3-T_{cf}}{R_{cf}}$$

From above Equation, we get:

$$T_{hf}-T_1=QR_{hf};\ T_1-T_2=QR_1;\ T_2-T_3=QR_2;\ T_3-T_{cf}=QR_{cf}$$

By adding these expressions, we get: $T_{hf}-T_{cf}=Q[R_{hf}+R_1+R_2+R_{cf}]$

$$\therefore Q=\frac{T_{hf}-T_{cf}}{R_{hf}+R_1+R_2+R_{cf}}=\frac{\Delta T}{\sum R} \qquad \text{(i)}$$

Let U_i be the overall heat transfer coefficient based on inner area and U_o be the overall heat transfer coefficient based on outer area. The rate of heat transfer from the hot to cold fluid in terms of overall heat transfer coefficient can be written as below.

$$Q=U_iA_i(T_{hf}-T_{cf})=U_oA_o(T_{hf}-T_{cf}) \qquad \text{(ii)}$$

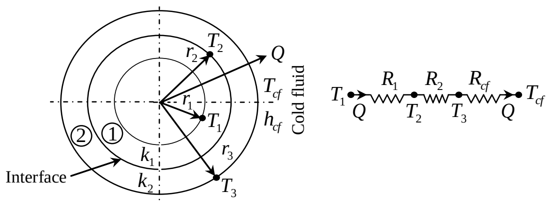
By comparing Equations (i) and (ii), we obtain:

$$U_i=\frac{1}{A_i\sum R}=\frac{1}{\left[\frac{1}{h_{hf}}+\frac{r_1^{2}}{k_1}\left(\frac{r_2-r_1}{r_1r_2}\right)+\frac{r_1^{2}}{k_2}\left(\frac{r_3-r_2}{r_2r_3}\right)+\frac{r_1^{2}}{r_3^{2}}\frac{1}{h_{cf}}\right]}$$

and $$U_o=\frac{1}{A_o\sum R}=\frac{1}{\left[\frac{r_3^{2}}{r_1^{2}}\frac{1}{h_{hf}}+\frac{r_3^{2}}{k_1}\left(\frac{r_2-r_1}{r_1r_2}\right)+\frac{r_3^{2}}{k_2}\left(\frac{r_3-r_2}{r_2r_3}\right)+\frac{1}{h_{cf}}\right]}$$

Question 34. A hollow spherical tank is made up of two materials; first with $k=80\,W/mK$ is having inner radius of 50 mm and outer radius of 150 mm and the second with $k=15\,W/mK$ forms the outer layer having radius of 200 mm. The outer surface is exposed to cold fluid at 293 K with $h=10\,W/m^2K$. If inner surface temperature is maintained at 570 K and there is perfect contact between two layers, evaluate (i) the steady rate of heat flow through this tank, (ii) interface temperature and (iii) overall heat transfer coefficient based on outer area.

Solution: Refer below Figure. Let $k_1=80\,W/mK$, $r_1=50\,mm=0.05\,m$, $r_2=150\,mm=0.15\,m$, $k_2=15\,W/mK$, $r_3=200\,mm=0.2\,m$, $T_{cf}=293\,K$, $h=10\,W/m^2K$ and $T_1=570\,K$.



$$R_1 = \frac{r_2 - r_1}{4\pi k_1 r_1 r_2} = \frac{0.15 - 0.05}{4\pi \times 80 \times 0.05 \times 0.15} = 0.013263 K/W$$

$$R_2 = \frac{r_3 - r_2}{4\pi k_2 r_2 r_3} = \frac{0.2 - 0.15}{4\pi \times 15 \times 0.15 \times 0.2} = 8.842 \times 10^{-3} \ K/W$$

$$R_{cf} = \frac{1}{A_o h_{cf}} = \frac{1}{4\pi r_3^2 h_{cf}} = \frac{1}{4\pi \times 0.2^2 \times 10} = 0.198944 K/W$$

$$\therefore \sum R = 0.013263 + 8.842 \times 10^{-3} + 0.198944 = 0.221049 K/W$$

(i) The steady rate of heat loss from spherical tank can be evaluated as follows.

$$Q = \frac{T_1 - T_{cf}}{\sum R} = \frac{570 - 293}{0.221049} = 1253.116 W$$ **(Ans.)**

(ii) The steady rate of heat transfer through the tank can be given as below.

$$Q = \frac{T_1 - T_2}{R_1}$$

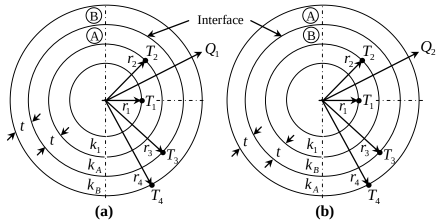
$$\therefore T_2 = T_1 - Q \times R_1 = 570 - 1253.116 \times 0.013263 = 553.4 \ K$$ **(Ans.)**

(iii) The overall heat transfer coefficient based on outer area can be evaluated as below.

$$U_o = \frac{1}{A_o \sum R} = \frac{1}{4\pi r_3^2 \sum R} = \frac{1}{4\pi \times 0.2^2 \times 0.221049} = 8.999 W/m^2 K$$ **(Ans.)**

Question 35. A hollow spherical metallic container $(k = 55 W/mK)$ of inner radius 5 cm and outer radius 6 cm is to be covered with two layers of insulating materials 'A' and 'B' each having a thickness of 4 cm. The thermal conductivity of insulating material 'A' next to the container is $0.08 W/mK$ and that of 'B' is $0.16 W/mK$. Evaluate the steady rate of heat loss and the interface temperature between the two layers of insulation when the temperatures of the inner surface of the container and outside surface of the insulation are 525 K and 310 K, respectively. If the orders of insulating material for the container were reversed, determine the percentage change in heat loss with all other conditions remain unchanged. Give your comment.

Solution Refer below Figure. Let $k_1 = 55 W/mK$, $r_1 = 5\,cm = 0.05\,m$, $r_2 = 6\,cm = 0.06\,m$, $t_A = t_B = t = 4\,cm = 0.04\,m$, $k_A = 0.08 W/mK$, $k_B = 0.16 W/mK$, $T_1 = 525 K$ and $T_4 = 310 K$.



$$r_3 = r_2 + t = 0.06 + 0.04 = 0.1\,m;\ r_4 = r_3 + t = 0.1 + 0.04 = 0.14\,m$$

Case I: When layer of insulating material 'A' is kept inside. Refer Figure (a).

Since $$Q_1 = \frac{4\pi(T_1 - T_4)}{\left[\frac{r_2 - r_1}{k_1 r_1 r_2} + \frac{r_3 - r_2}{k_A r_2 r_3} + \frac{r_4 - r_3}{k_B r_3 r_4}\right]}$$

Thus $$Q_1 = \frac{4\pi \times (525 - 310)}{\left[\frac{0.06 - 0.05}{55 \times 0.05 \times 0.06} + \frac{0.1 - 0.06}{0.08 \times 0.06 \times 0.1} + \frac{0.14 - 0.1}{0.16 \times 0.1 \times 0.14}\right]} = 26.684\,W$$

The interface temperature can be obtained as follows.

$$Q_1 = \frac{4\pi k_B r_3 r_4 (T_3 - T_4)}{r_4 - r_3} \Rightarrow T_3 = \frac{Q_1 (r_4 - r_3)}{4\pi k_B r_3 r_4} + T_4$$

$$\therefore T_3 = \frac{26.684 \times (0.14 - 0.1)}{4\pi \times 0.16 \times 0.1 \times 0.14} + 310 = 347.92\,K$$ **(Ans.)**

Case II: When layer of insulating material 'A' is kept outside. Refer Figure (b).

Since $$Q_2 = \frac{4\pi(T_1 - T_4)}{\left[\frac{r_2 - r_1}{k_1 r_1 r_2} + \frac{r_3 - r_2}{k_B r_2 r_3} + \frac{r_4 - r_3}{k_A r_3 r_4}\right]}$$

Thus $$Q_2 = \frac{4\pi \times (525 - 310)}{\left[\frac{0.06 - 0.05}{55 \times 0.05 \times 0.06} + \frac{0.1 - 0.06}{0.16 \times 0.06 \times 0.1} + \frac{0.14 - 0.1}{0.08 \times 0.1 \times 0.14}\right]} = 34.89\,W$$ **(Ans.)**

Since $Q_2 > Q_1$, therefore, layer of insulating material 'A' next to the container decreases the heat transfer rate. The percentage decrease in heat transfer rate is obtained as below.

$$\frac{Q_2 - Q_1}{Q_2} \times 100 = \frac{34.89 - 26.684}{34.89} \times 100 = 23.52\%$$ **(Ans.)**

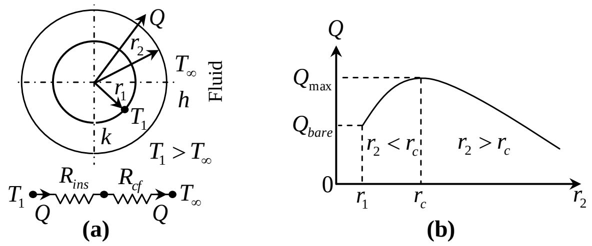
Comment: For the insulation of a spherical container, material with lower thermal conductivity should be applied next to the hot surface.

Question 36. What do you mean by critical radius of insulation? Give examples where this concept is applied.

Answer: Addition of insulation to a plane wall always decreases heat transfer. Since, heat transfer area remains constant, thus any addition of insulation always increases the thermal resistance of the wall without increasing the convection resistance. But in the case of a cylinder or sphere the addition of insulation increases the conduction resistance but decreases the convection resistance due to the increase in surface area for convection. The heat transfer from a cylindrical or spherical wall may increase or decrease, depending on which effect dominates. These two opposing effects lead to critical thickness of insulation (or an optimum thickness of insulation). The most practical examples of such cases are the wires carrying electric currents which generate heat, steam pipes carrying steam from a boiler, cylindrical and spherical storage tanks containing hot fluids.

Question 37. Derive an expression for the critical radius of insulation for cylinders. How it is useful in cable industry?

Answer: Consider a cylindrical pipe surrounded by an insulating layer of thermal conductivity k. Let r_1 and r_2 be the inner and outer radii of insulation, respectively. Let us assume that the inner surface of insulation is maintained at a temperature of T_1 and outer surface is convecting heat to a fluid of temperature T_∞ with a convective heat transfer coefficient of h as shown in below Figure (a).



The steady heat transfer rate (Q) through the cylindrical insulation layer is given by:

$$Q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_{ins} + R_{cf}} = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi kL} + \dfrac{1}{2\pi r_2 Lh}}$$

Here, L is the length of the pipe, R_{ins} is the conductive thermal resistance of insulation and R_{cf} is the convective thermal resistance to heat flow.

When T_1, T_∞, r_1, k and h have been fixed, the steady rate of heat transfer Q depends only on r_2. As r_2 increases, R_{ins} increases but R_{cf} decreases. Therefore, Q is expected to have either a maximum or a minimum in relation to its variation with r_2. Maximum heat transfer rate is obtained when thermal resistance is minimum. This can be checked mathematically from the values of $d\left(\sum R\right)/dr_2$ and $d^2\left(\sum R\right)/dr_2^{\,2}$.

Differentiating the total thermal resistance $\left(\sum R\right)$ with respect to r_2 as follows.

$$\frac{d\left(\sum R\right)}{dr_2} = \frac{d}{dr_2}\left[\frac{\ln(r_2/r_1)}{2\pi kL} + \frac{1}{2\pi r_2 Lh}\right] = \frac{d}{dr_2}\left[\frac{1}{2\pi kL}\ln r_2 - \frac{1}{2\pi kL}\ln r_1 + \frac{1}{2\pi r_2 Lh}\right]$$

Thus $$\frac{d\left(\sum R\right)}{dr_2} = \frac{1}{2\pi kL} \times \frac{1}{r_2} - \frac{1}{2\pi Lh} \times \frac{1}{r_2^{\,2}} \qquad \text{(i)}$$

By putting $(d\sum R/dr_2) = 0$ as the condition for either minimum or maximum, we get:

$$\frac{1}{2\pi kL}\times\frac{1}{r_2}-\frac{1}{2\pi Lh}\times\frac{1}{r_2^{\,2}}=0\Rightarrow\frac{1}{2\pi kL}\times\frac{1}{r_2}=\frac{1}{2\pi Lh}\times\frac{1}{r_2^{\,2}}$$

$$\therefore r_2=\frac{k}{h}$$

In order to determine whether the above result minimizes or maximizes the total resistance, the second derivative is to be evaluated by differentiating Equation (i) as,

$$\frac{d^2\left(\sum R\right)}{dr_2^{\,2}}=\frac{d^2}{dr_2^{\,2}}\left[\frac{1}{2\pi kL}\times\frac{1}{r_2}-\frac{1}{2\pi Lh}\times\frac{1}{r_2^{\,2}}\right]=-\frac{1}{2\pi kL}\times\frac{1}{r_2^{\,2}}+\frac{1}{\pi Lh}\times\frac{1}{r_2^{\,3}}$$

Substituting $r_2=(k/h)$ in the above equation, we obtain the below result.

$$\frac{d^2\left(\sum R\right)}{dr_2^{\,2}}=-\frac{1}{2\pi kL}\times\frac{1}{(k/h)^2}+\frac{1}{\pi Lh}\times\frac{1}{(k/h)^3}=\frac{h^3}{2\pi k^3L}$$

The second derivative is a positive quantity verifying that $r_2=(k/h)$ represents the condition for minimum resistance and consequently, gives the maximum heat flow rate. The value of $r_2=(k/h)$ is called as the critical radius of insulation which is denoted by r_c. Therefore, critical radius of insulation for a cylinder (or pipe) can be given by the following equation.

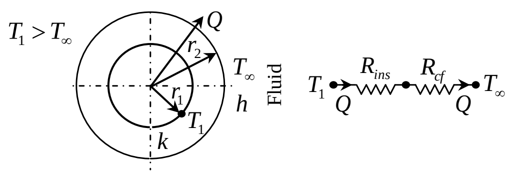
$$r_c=\frac{k}{h}$$

The thickness of insulation layer given by (r_c-r_1) is called the critical thickness of insulation, at which heat loss is maximum $(Q_{\max})$. The variation of Q with outer radius of insulation r_2 is illustrated in Figure (b). It can be seen that if $r_2<r_c$ and insulation is added to the pipe (or cylinder) heat loss increases and it reaches a maximum at r_c and starts to decrease for $r_2>r_c$.

This concept of critical radius of insulation is used in electric cable industries. The insulation is provided to current carrying electric wires such that the outer radius should be less than r_c or equal to r_c to dissipate maximum amount of heat. This insulation on electric wires not only provides safety against some grounded surface but also keeps the operating temperature of wire or cable steady and within safety limits by dissipating heat at the same rate at which it is generated.

Question 38. Derive an expression for the critical radius of insulation for spheres.

Answer: The steady rate of heat transfer (Q) through the insulation layer provided over a sphere as shown in below Figure is given as,



$$Q=\frac{T_1-T_\infty}{\sum R}=\frac{T_1-T_\infty}{R_{ins}+R_{cf}}=\frac{T_1-T_\infty}{\frac{r_2-r_1}{4\pi k r_1 r_2}+\frac{1}{4\pi r_2^{\,2}h}}$$

We know that for maximum heat flow rate, $\sum R$ should be minimum.

$$\therefore \frac{d\left(\sum R\right)}{dr_2}=\frac{d}{dr_2}\left[\frac{r_2-r_1}{4\pi k r_1 r_2}+\frac{1}{4\pi r_2^{\,2}h}\right]=0 \text{ or } \frac{d}{dr_2}\left[\frac{r_2}{4\pi k r_1 r_2}-\frac{r_1}{4\pi k r_1 r_2}+\frac{1}{4\pi r_2^{\,2}h}\right]=0$$

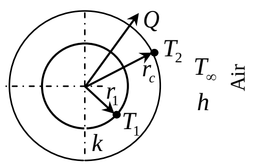
$$\frac{1}{4\pi k r_2^{\,2}}-\frac{2}{4\pi r_2^{\,3}h}=0 \text{ or } \frac{1}{4\pi k r_2^{\,2}}=\frac{2}{4\pi r_2^{\,3}h}$$

$$\therefore r_2=\frac{2k}{h}=r_c$$

Above Equation gives the critical radius (r_c) for a sphere, therefore, the critical thickness of insulation for a sphere $=(r_c-r_1)$.

Question 39. A steam pipe covered with insulating material $(k=0.8\,W/mK)$ has inner and outer diameters as 100 mm and 110 mm, respectively. The inner surface and the surrounding air temperatures are 475 K and 295 K, respectively and the external convective heat transfer coefficient is $8\,W/m^2K$. Determine (i) the critical radius of insulation, (ii) the critical thickness of insulation, (iii) the rate of heat loss from the pipe at the critical radius of insulation for unit length and (iv) the outer surface temperature.

Answer: Refer below Figure. Let $k=0.8\,W/mK$, $d_1=100\,mm$, $d_2=110\,mm$, $T_1=475\,K$, $T_\infty=295\,K$, $h=8\,W/m^2K$ and $L=1\,m$.



$$r_1=d_1/2=100/2=50\,mm=0.05\,m;\; r_2=d_2/2=110/2=55\,mm=0.055\,m$$

(i) The critical radius of insulation for a pipe can be obtained as follows.

$$r_c=\frac{k}{h}=\frac{0.8}{8}=0.1\,m$$ **(Ans.)**

(ii) The critical thickness of insulation is given as below.

$$r_c-r_1=0.1-0.05=0.05\,m \text{ or } 50\,mm$$ **(Ans.)**

(iii) The rate of heat loss from the pipe at the critical radius of insulation is obtained as,

$$Q=\frac{2\pi L(T_1-T_\infty)}{\frac{\ln(r_c/r_1)}{k}+\frac{1}{r_c h}}=\frac{2\pi\times1\times(475-295)}{\frac{\ln(0.1/0.05)}{0.8}+\frac{1}{0.1\times8}}=534.38\,W$$ **(Ans.)**

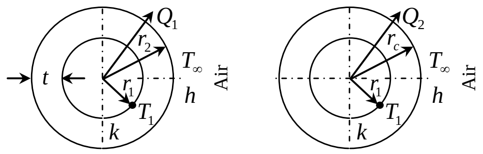
(iv) The outer surface temperature (T_2) can be calculated from the following relation,

$$Q = hA_o(T_2 - T_\infty) = h \times 2\pi r_c L \times (T_2 - T_\infty)$$

$$\therefore T_2 = \frac{Q}{2\pi r_c L h} + T_\infty = \frac{534.38}{2\pi \times 0.1 \times 1 \times 8} + 295 = 401.3\,K$$ **(Ans.)**

Question 40. A 2 mm diameter wire with 1 mm thick layer of insulation $(k = 0.2\,W/mK)$ used in an appliance is exposed to surrounding atmosphere $(h = 40\,W/m^2K)$. Calculate critical radius of insulation. What percentage change in heat transfer rate would occur if critical thickness of insulation is applied? Assume that temperature difference between surface of the wire and the surrounding air remains unchanged.

Solution Refer below Figure. Let $d_1 = 2\,mm$, $t = 1\,mm$, $k = 0.2\,W/mK$, $h = 40\,W/m^2K$ and $T_1 - T_\infty = \Delta T$.



(a) (b)

$$r_1 = d_1/2 = 2/2 = 1\,mm = 0.001\,m; \quad r_2 = r_1 + t = 1 + 1 = 2\,mm = 0.002\,m$$

The critical radius of insulation for a wire can be obtained as,

$$r_c = \frac{k}{h} = \frac{0.2}{40} = 0.005\,m$$ **(Ans.)**

Case I: Refer Figure (a). Heat transfer rate through the wire is obtained as,

$$Q_1 = \frac{2\pi L \Delta T}{\frac{\ln(r_2/r_1)}{k} + \frac{1}{r_2 h}} = \frac{2\pi L \Delta T}{\frac{\ln(0.002/0.001)}{0.2} + \frac{1}{0.002 \times 40}} = 0.393542 L\Delta T$$

Case II: Refer Figure (b). Heat transfer rate through the wire when critical thickness of insulation is applied,

$$Q_2 = \frac{2\pi L \Delta T}{\frac{\ln(r_c/r_1)}{k} + \frac{1}{r_c h}} = \frac{2\pi L \Delta T}{\frac{\ln(0.005/0.001)}{0.2} + \frac{1}{0.005 \times 40}} = 0.481574 L\Delta T$$

Since $Q_2 > Q_1$, therefore, percentage increase in heat transfer rate when critical thickness of insulation is applied can be obtained as,

$$\frac{Q_2 - Q_1}{Q_1} \times 100 = \frac{0.481574 - 0.393542}{0.393542} \times 100 = 22.37\%$$ **(Ans.)**

CHAPTER - 3

STEADY STATE HEAT CONDUCTION WITH HEAT GENERATION

Question 1. Define internal heat generation. Also give practical cases where it is used.

Answer: The conversion of electrical, nuclear and chemical energies inside the body is called as internal heat generation. It is a volumetric phenomenon, therefore, the rate of heat generation (q_g) in a medium is generally specified per unit volume and its units are W/m^3. Heat generation occurs within the system (or body) such as resistance heaters, nuclear reactors, chemical and combustion process, drying and setting of concrete.

Question 2. Give the general solution for temperature distribution for a plane wall made of homogeneous and isotropic material with uniform heat generation for 1-D heat conduction (i.e., x-direction) under steady state.

Answer: The governing equation for heat conduction in one direction (i.e., x-direction) under steady state through a plane wall made of homogeneous and isotropic material with uniform heat generation can be given by as,

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0$$

or $$\frac{d^2T}{dx^2} = -\frac{q_g}{k}$$

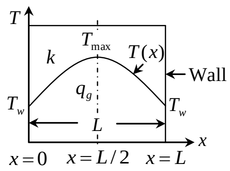
Integrating, we get: $\dfrac{dT}{dx} = -\dfrac{q_g}{k}x + C_1$

Integrating again, we get: $T = -\dfrac{q_g}{2k}x^2 + C_1 x + C_2$

The above equation gives the general solution for temperature distribution in a plane wall with uniform heat generation in which C_1 and C_2 are arbitrary constants whose values can be obtained with the help of the given boundary conditions.

Question 3. In one-dimensional heat transfer, derive expressions for temperature distribution within the plane wall and prove that the heat generated is equal to the heat loss from the sides when this wall is subjected to uniform heat generation and exposed to a fluid at the same temperature on both sides.

Answer: Consider a plane wall of thickness L with uniform internal heat generation q_g in W/m^3 and constant thermal conductivity k. Let the dimensions of this wall in y- and z-directions are comparatively large, so that heat transfer takes place in x-direction only and temperatures on the two faces of the wall are maintained at T_w as shown in below Figure.



The general solution for temperature distribution for a plane wall with uniform heat generation is as,

$$T=-\frac{q_g}{2k}x^2+C_1x+C_2 \qquad \text{(i)}$$

Now applying boundary conditions: (i) $T=T_w$ at $x=0$ and (ii) $T=T_w$ at $x=L$.

Substituting first boundary condition (b. c.), we get:

$$T_w=-\frac{q_g}{2k}(0)^2+C_1(0)+C_2 \Rightarrow \therefore C_2=T_w$$

Now substituting second b. c. and value of $C_2=T_w$ in Equation (i), we get:

$$T_w=-\frac{q_g}{2k}(L)^2+C_1(L)+T_w \Rightarrow \therefore C_1=\frac{q_g}{2k}L$$

Substituting the values of C_1 and C_2 in Equation (i), we get:

$$T=-\frac{q_g}{2k}x^2+\frac{q_g}{2k}Lx+T_w$$

$$\therefore T=\frac{q_g}{2k}(L-x)x+T_w \qquad \text{(ii)}$$

Now $\quad \dfrac{dT}{dx}=\dfrac{q_g}{2k}(L-2x)$

Put $(dT/dx)=0$ as the condition for either maximum or minimum, we have:

$$\frac{q_g}{2k}(L-2x)=0$$

$$L-2x=0 \qquad (\because q_g/2k\neq 0)$$

$$\therefore x=\frac{L}{2}$$

And $\quad \dfrac{d^2T}{dx^2}=-\dfrac{q_g}{k}$

The second derivative is a negative quantity, therefore, T has a maximum at $x=L/2$. Equation (ii) is quadratic thus the temperature distribution is parabolic in nature and symmetrical about the centre plane. By substituting $x=L/2$ in Equation (ii) we get the maximum value of temperature as below.

$$T_{max} = \frac{q_g}{2k}\left(L - \frac{L}{2}\right)\frac{L}{2} + T_w = \frac{q_g}{8k}L^2 + T_w \qquad \text{(iii)}$$

Heat transfer takes place from both the surfaces (i.e., $x = 0$ and $x = L$) and is equal. The heat transfer rate for each surface is given as follows.

$$Q = -kA\left(\frac{dT}{dx}\right)_{x=0\ or\ x=L} = -kA\left[\frac{q_g}{2k}(L - 2x)\right]_{x=0\ or\ x=L} = \frac{AL}{2}q_g \qquad \text{(iv)}$$

Here, A is the surface area of the wall normal to the direction of heat flow.

When both surfaces are considered, the heat transfer rate is given as follows.

$$Q = 2 \times \frac{AL}{2}q_g = AL \times q_g = volume \times heat\ generation rate \qquad \text{(v)}$$

Heat conducted to the wall surface is convected away to the surrounding fluid (or air) at temperature T_a with heat transfer coefficient h. Thus from energy balance at each surface, we get:

$$\frac{AL}{2}q_g = hA(T_w - T_a)$$

Thus $\quad T_w = T_a + \frac{q_g}{2h}L$

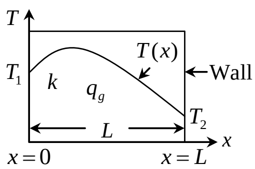
Substituting T_w in Equation (ii) and (iii), we get:

$$T = T_a + \frac{q_g}{2h}L + \frac{q_g}{2k}(L - x)x$$

$$T_{max} = T_a + \frac{q_g}{2h}L + \frac{q_g}{8k}L^2 = T_a + q_g\left(\frac{L}{2h} + \frac{L^2}{8k}\right)$$

Question 4. In one-dimensional heat transfer, derive expressions for temperature distribution within the plane wall and rate of heat transfer when the surfaces of this wall are subjected to uniform heat generation and exposed to fluids at the different temperatures.

Answer: Consider a plane wall of thickness L with uniform internal heat generation q_g in W/m^3 and constant thermal conductivity k. Let the heat transfer takes place in x-direction only and the two faces of the wall are maintained at a different temperatures T_1 and T_2 as shown in below Figure.



Below Equation (i) is the general solution for temperature distribution for a plane wall with uniform heat generation,

$$T = -\frac{q_g}{2k}x^2 + C_1 x + C_2 \qquad \text{(i)}$$

The boundary conditions are: (i) $T = T_1$ at $x = 0$ and (ii) $T = T_2$ at $x = L$.

By applying these boundary conditions to Equation (i), we get:

$$C_2 = T_1 \text{ and } C_1 = \frac{T_2 - T_1}{L} + \frac{q_g}{2k}L$$

Substituting the values of C_1 and C_2 in Equation (i), we get:

$$T = -\frac{q_g}{2k}x^2 + \left(\frac{T_2 - T_1}{L} + \frac{q_g}{2k}L\right)x + T_1$$

or $$T = \left[\frac{q_g}{2k}(L - x) + \frac{(T_2 - T_1)}{L}\right]x + T_1 \qquad \text{(ii)}$$

In order to find where the maximum temperature occurs, differentiate Equation (ii) with respect to x and equate to zero. Then put this value of x in Equation (ii) to get the value of maximum temperature $(T_{\max})$. If maximum temperature lies inside the wall then heat will transfer to the both surfaces. Total heat generated in the wall is, $Q = q_g AL$. Part of this heat generated flows to the left surface and remaining part flows to the right surface.

Differentiating Equation (ii), we get:

$$\frac{dT}{dx} = \frac{q_g}{2k}(L - 2x) + \frac{(T_2 - T_1)}{L}$$

The rate of heat flow from the surfaces at $x = 0$ is given as below.

$$Q_1 = -kA\left(\frac{dT}{dx}\right)_{x=0} = -kA\left[\frac{q_g}{2k}(L - 2x) + \frac{(T_2 - T_1)}{L}\right]_{x=0} = kA\left[\frac{T_1 - T_2}{L} - \frac{q_g L}{2k}\right] \qquad \text{(iii)}$$

The rate of heat flow from the surfaces at $x = L$ is given as below.

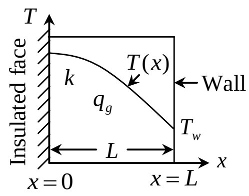
$$Q_2 = -kA\left(\frac{dT}{dx}\right)_{x=L} = -kA\left[\frac{q_g}{2k}(L - 2x) + \frac{(T_2 - T_1)}{L}\right]_{x=L} = kA\left[\frac{T_1 - T_2}{L} + \frac{q_g L}{2k}\right] \qquad \text{(iv)}$$

Thus total heat transfer will be equal to the sum of Q_1 and Q_2.

If maximum temperature is T_1 (i.e., at $x = 0$) then the heat transfer will be in the direction of increasing x and the total heat transfer will be equal to Q_2 only. In Equation (iv), if $q_g = 0$ then, it becomes the equation for plane wall without heat generation.

Question 5. In one-dimensional heat transfer, derive expression for temperature distribution within the plane wall when one of the surfaces is exposed to fluid and other surface is insulated.

Answer: Consider a plane wall of thickness L with uniform internal heat generation q_g in W/m^3 and constant thermal conductivity k. Let one surface of this wall $(x=0)$ is perfectly insulated and the other surface $(x=L)$ is maintained at a uniform temperature T_w as shown in below Figure.



Below Equation (i) is the general solution for temperature distribution for a plane wall with uniform heat generation,

$$T = -\frac{q_g}{2k}x^2 + C_1 x + C_2 \qquad \text{(i)}$$

The boundary conditions are: (i) $(dT/dx) = 0$ at $x = 0$ and (ii) $T = T_w$ at $x = L$.

By applying these boundary conditions to Equation (i), we get:

$$C_1 = 0 \text{ and } C_2 = T_w + \frac{q_g}{k}.\frac{L^2}{2}$$

Substituting the values of C_1 and C_2 in Equation (i), we get:

$$T = -\frac{q_g}{k}.\frac{x^2}{2} + T_w + \frac{q_g}{k}.\frac{L^2}{2} \qquad \text{(ii)}$$

Now $$\frac{dT}{dx} = -\frac{q_g x}{k}$$

By putting $(dT/dx) = 0$ as the condition for either maximum or minimum, we have:

$$-\frac{q_g x}{k} = 0$$

$$\therefore x = 0$$

And $$\frac{d^2T}{dx^2} = -\frac{q_g}{k}$$

The second derivative is a negative quantity, therefore, T has a maximum at $x = 0$.

Maximum temperature is given by substituting $x = 0$ in Equation (ii) as,

$$T_{\max} = T_w + \frac{q_g}{k}.\frac{L^2}{2} \qquad \text{(iii)}$$

When heat conducted to the wall surface is convected away to the surrounding fluid at temperature T_a with heat transfer coefficient h, from energy balance surface, we get:

$$q_g AL = hA(T_w - T_a)$$

$$\therefore T_w = T_a + \frac{q_g L}{h} \qquad \text{(iv)}$$

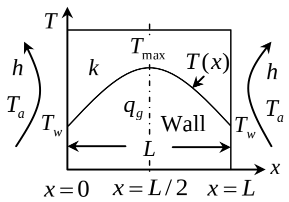
Substituting T_w in Equations (ii) and (iii), we get:

$$T = T_a + \frac{q_g L}{h} - \frac{q_g}{k}.\frac{x^2}{2} + \frac{q_g}{k}.\frac{L^2}{2}$$

$$T_{\max} = T_a + \frac{q_g L}{h} + \frac{q_g}{k}.\frac{L^2}{2}$$

Question 6. The internal heat generation due to passage of electric current in a 0.1 m thick plate $(k = 20\, W/mK)$ is $6.4\times10^4\, W/m^3$. If the air temperature is 295 K and the convective heat transfer coefficient is $25\, W/m^2K$, determine (i) surface temperature and (ii) maximum temperature in the plate.

Solution: Refer below Figure. Let $L = 0.1\, m$, $k = 20\, W/mK$, $q_g = 6.4\times10^4\, W/m^3$, $T_a = 295\, K$ and $h = 25\, W/m^2K$.



(i) The surface temperature can be determined as,

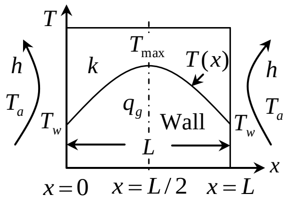
$$T_w = T_a + \frac{q_g}{2h} L = 295 + \frac{6.4\times10^4}{2\times25}\times0.1 = 423\, K \ \textbf{(Ans.)}$$

(ii) The maximum temperature can be determined as,

$$T_{\max} = \frac{q_g}{8k} L^2 + T_w = \frac{6.4\times10^4}{8\times20}\times(0.1)^2 + 423 = 427\, K \ \textbf{(Ans.)}$$

Question 7. A heat generating plate of 2 cm thick and 10 cm wide with $k = 25\, W/mK$ is used to heat a fluid at 303 K. The heat generation rate in the plate is $7.5\times10^5\, W/m^3$ due to passage of electric current through it. Determine the convective heat transfer coefficient to maintain the temperature of the plate below 453 K. Neglect the heat loss from the edges of the plate.

Solution: Refer below Figure. Let $L = 2\, cm = 0.02\, m$, $b = 10\, cm = 0.1\, m$, $k = 25\, W/mK$, $T_a = 303\, K$, $q_g = 7.5\times10^5\, W/m^3$, $T_{\max} = 453\, K$ and h be the convective heat transfer coefficient.



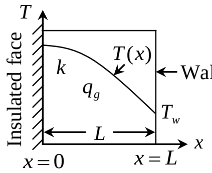
Since $T_{max} = T_a + q_g\left(\frac{L}{2h} + \frac{L^2}{8k}\right)$

Thus $453 = 303 + 7.5\times10^5 \times\left(\frac{0.02}{2\times h} + \frac{0.02^2}{8\times25}\right)$

$\therefore h = 50.5\, W/m^2K$ **(Ans.)**

Question 8. A plane wall 0.8 m thick with thermal conductivity of $15\,W/mK$ is well insulated on its left side while the right side surface is maintained at a uniform temperature of 300°C. It generates heat uniformly at a rate of 500 W/m³ when an electric current is passed through it. Determine the maximum temperature in the wall and the location of the plane where it occurs.

Solution: Refer below Figure. Let $L = 0.8\,m$, $k = 15\,W/mK$, $T_w = 300\,^oC = 573\,K$ and $q_g = 500\,W/m^3$.



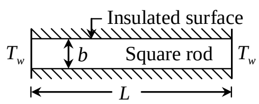
Maximum temperature (T_{max}) can be obtained by using below Equation as,

$$T_{max} = T_w + \frac{q_g}{k}.\frac{L^2}{2} = 573 + \frac{500}{15}\times\frac{0.8^2}{2} = 585.8\,K \; or \; 312.8\,^oC$$ **(Ans.)**

T_{max} will occur at the insulated face, i.e., at $x = 0$. **(Ans.)**

Question 9. A square cross-section aluminium rod of 4.5 mm side and 1 m length is used to carry current between two bus bars which are maintained at 300 K. The lateral surface area of the rod is well insulated. The specific resistance and the thermal conductivity of the rod are $2.65\times10^{-8}\,\Omega m$ and $237\,W/mK$, respectively. Determine the maximum current the rod can carry if its temperature is not to exceed 450 K.

Solution: Refer below Figure. Let $b = 4.5\,mm = 0.0045\,m$, $L = 1\,m$, $T_w = 300\,K$, $\rho = 2.65\times10^{-8}\,\Omega m$, $k = 237\,W/mK$ and $T_{max} = 450\,K$.



Maximum temperature for 1-D heat flow can be given by below Equation as,

$$T_{max} = \frac{q_g}{8k}L^2 + T_w \quad \text{or} \quad 450 = \frac{q_g}{8\times 237}\times 1^2 + 300$$

$$\therefore q_g = (450-300)\times 8\times 237 = 284400 W/m^3$$

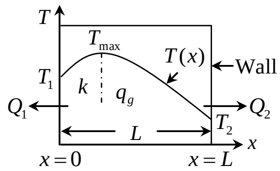
$$Q_g = q_g \times volume = q_g \times A \times L = q_g \times b^2 \times L = 284400\times 0.0045^2 \times 1 = 5.7591 W$$

Also $$Q_g = I^2 \times R_e = I^2 \times \frac{\rho L}{A} \quad \text{or} \quad 5.7591 = I^2 \times \frac{2.65\times 10^{-8}\times 1}{0.0045^2}$$

$$\therefore I = \sqrt{\frac{5.7591\times 0.0045^2}{2.65\times 10^{-8}}} = 66.34\, A$$ **(Ans.)**

Question 10. A 3 cm thick cast iron plate $(k = 55 W/mK)$ has a uniform internal heat generation of $3.3\times 10^7\ W/m^3$. If end effects are negligible and the temperatures on the surfaces of this plate are 435 K and 375 K, determine (i) the temperature distribution across the plate, (ii) the value and position of the maximum temperature and (iii) heat flow per unit area from each surface of the plate.

Solution: Refer below Figure. Let $L = 3\,cm = 0.03\,m$, $k = 55 W/mK$, $q_g = 3.3\times 10^7\ W/m^3$, $T_1 = 435 K$, $T_2 = 375 K$ and $A = 1\, m^2$.



(i) $$T = \left[\frac{q_g}{2k}(L-x) + \frac{(T_2 - T_1)}{L}\right]x + T_1$$

$$T = \left[\frac{3.3\times 10^7}{2\times 55}\times(0.03 - x) + \frac{(375-435)}{0.03}\right]x + 435$$

$$\therefore T = 435 + 7000x - 300000x^2$$ **(Ans.)**

(ii) $$\frac{dT}{dx} = 7000 - 600000x$$

For determining the position of maximum temperature put $(dT/dx) = 0$.

Thus $$7000 - 600000x = 0$$

$$x = \frac{7000}{600000} = 0.0117\, m\ or\ 1.17\, cm$$ **(Ans.)**

$$\therefore T_{max} = 435 + 7000\times 0.0117 - 300000\times 0.0117^2 = 475.83 K$$ **(Ans.)**

(iii) The heat flux at the left face $(x = 0)$ is given as below.

$$Q_1 = -kA\left(\frac{dT}{dx}\right)_{x=0} = -55\times1\times(7000-600000x)_{x=0} = -3.85\times10^5\ W$$ **(Ans.)**

The negative sign indicates that heat flow at the left face is in a direction opposite to that of measurement of distance, i.e., negative x -direction.

The heat flux at the right face $(x = L = 0.03\,m)$ is given as below.

$$Q_2 = -kA\left(\frac{dT}{dx}\right)_{x=0.03} = -55\times1\times(7000-600000x)_{x=0.03} = 6.05\times10^5\ W$$ **(Ans.)**

The positive sign indicates that heat flow at the right face is in the positive x -direction.

Check: The sum of Q_1 and Q_2 which is equal to $9.9\times10^5\ W$ must be equal the total heat generated per unit area of the plate.

$$Q_g = q_g \times A \times L = 3.3\times10^7\times1\times0.03 = 9.9\times10^5\ W$$

Question 11. Give the general solution for temperature distribution for a cylinder made of homogeneous and isotropic material with uniform heat generation for 1-D heat conduction (i.e., r -direction) under steady state.

Answer: The governing equation for heat conduction in one direction (r -direction) under steady state through cylindrical bodies made of homogeneous and isotropic material with uniform heat generation can be given by below Equation as,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{q_g}{k} = 0$$

or $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{q_g r}{k}$$

Integrating, we get: $r\dfrac{dT}{dr} = -\dfrac{q_g r^2}{2k} + C_1$

or $$\frac{dT}{dr} = -\frac{q_g r}{2k} + \frac{C_1}{r}$$

Integrating again, we get: $T = -\dfrac{q_g r^2}{4k} + C_1 \ln r + C_2$

The above equation gives the general solution for temperature distribution in a cylinder with uniform heat generation in which C_1 and C_2 are arbitrary constants whose values can be obtained with the help of the given boundary conditions.

Question 12. Derive the expressions for temperature distribution and rate of heat conduction for a hollow cylinder made of homogeneous and isotropic material with uniform heat generation for 1-D heat conduction (i.e., r -direction) under steady state whose both surfaces maintained at constant temperatures.

Answer: Consider a hollow cylinder of length L with a uniform internal heat generation rate of q_g in W/m^3. Let r_1 be the inner radius, r_2 be the outer radius, T_1 be the inner surface temperature, T_2 be outer surface temperature and k be the constant thermal conductivity.

The temperature distribution in a cylinder is given by below Equation as,

$$T = -\frac{q_g r^2}{4k} + C_1 \ln r + C_2 \qquad \text{(i)}$$

$$\frac{dT}{dr} = -\frac{q_g r}{2k} + \frac{C_1}{r}$$

The constants of integration can be determined by applying the boundary conditions: (i) $T = T_1$ at $r = r_1$ and (ii) $T = T_2$ at $r = r_2$ as follows.

Applying first b.c. to Equation (i), we get:

$$T_1 = -\frac{q_g r_1^2}{4k} + C_1 \ln r_1 + C_2 \qquad \text{(ii)}$$

Applying second b.c. to Equation (i), we get:

$$T_2 = -\frac{q_g r_2^2}{4k} + C_1 \ln r_2 + C_2 \qquad \text{(iii)}$$

Solving equations (ii) and (iii), we get the values of C_1 and C_2 as below.

$$C_1 = \frac{(T_2 - T_1) + \frac{q_g}{4k}(r_2^2 - r_1^2)}{\ln(r_2/r_1)} \quad \text{and} \quad C_2 = T_1 + \frac{q_g r_1^2}{4k} - \frac{(T_2 - T_1) + \frac{q_g}{4k}(r_2^2 - r_1^2)}{\ln(r_2/r_1)}$$

Substituting these values in Equation (i), the temperature distribution becomes,

$$T = -\frac{q_g r^2}{4k} + \frac{(T_2 - T_1) + \frac{q_g}{4k}(r_2^2 - r_1^2)}{\ln(r_2/r_1)} \ln r + T_1 + \frac{q_g r_1^2}{4k} - \frac{(T_2 - T_1) + \frac{q_g}{4k}(r_2^2 - r_1^2)}{\ln(r_2/r_1)}$$

or $$\frac{T - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)} + \frac{q_g}{4k}\frac{(r_2^2 - r_1^2)}{(T_2 - T_1)}\left[\frac{\ln(r/r_1)}{\ln(r_2/r_1)} - \frac{(r/r_1)^2 - 1}{(r_2/r_1)^2 - 1}\right]$$

The rate of heat conduction through the cylinder is given as below.

$$Q = -kA\left(\frac{dT}{dr}\right) = -k \times (2\pi r L) \times \left(-\frac{q_g r}{2k} + \frac{C_1}{r}\right) = 2\pi L \times \left(-\frac{q_g r^2}{2k} + C_1 k\right)$$

Substituting value of C_1, we get:

$$Q = 2\pi L \times \left[-\frac{q_g r^2}{2k} + \frac{(T_2 - T_1) + \frac{q_g}{4k}(r_2{}^2 - r_1{}^2)}{\ln(r_2 / r_1)} . k \right]$$

Question 13. Derive the expressions for temperature distribution and rate of heat conduction for a solid cylinder made of homogeneous and isotropic material with uniform heat generation for 1-D heat conduction (i.e., r -direction) under steady state.

Answer: Consider a solid cylinder of radius R and length L with a uniform internal heat generation rate of q_g in W / m^3. The general solution for temperature distribution in a cylinder is given as,

$$T = -\frac{q_g r^2}{4k} + C_1 \ln r + C_2 \qquad \text{(i)}$$

The boundary conditions are:

(i) At $r = R$, i.e., at the surface, $T = T_s$ and

(ii) Heat generated = Heat lost by conduction at the surface of the cylinder

From the second boundary condition, we get:

$$q_g \times \pi R^2 L = -k \times 2\pi R L \times \left(\frac{dT}{dr} \right)_{r=R}$$

Thus $$\left(\frac{dT}{dr} \right)_{r=R} = -\frac{q_g R}{2k} \qquad \text{(ii)}$$

The temperature gradient at the surface can also be obtained by differentiating Equation (i) and replacing r by R as below.

$$\left(\frac{dT}{dr} \right)_{r=R} = -\frac{q_g R}{2k} + \frac{C_1}{R} \qquad \text{(iii)}$$

From the expressions (ii) and (iii), we get:

$$-\frac{q_g R}{2k} + \frac{C_1}{R} = -\frac{q_g R}{2k}$$

$$\therefore C_1 = 0$$

Using first boundary condition and value of C_1 in Equation (i), we get:

$$T_s = -\frac{q_g R^2}{4k} + (0)\ln R + C_2$$

$$\therefore C_2 = T_s + \frac{q_g R^2}{4k}$$

Substituting values of C_1 and C_2 in Equation (i), we get:

$$T = -\frac{q_g r^2}{4k} + (0)\ln r + T_s + \frac{q_g R^2}{4k}$$

$$\therefore T = T_s + \frac{q_g}{4k}(R^2 - r^2) \quad \text{(iv)}$$

$$\frac{dT}{dr} = -\frac{q_g r}{2k}$$

It can be observed from Equation (iv) that the temperature distribution is parabolic in nature and the maximum temperature $(T_{\max})$ occurs at the centre (i.e., at $r = 0$).

$$\therefore T_{\max} = T_s + \frac{q_g}{4k} R^2 \quad \text{(v)}$$

From Equations (iv) and (v), the dimensionless form of temperature distribution can be given as below.

$$\frac{T - T_s}{T_{\max} - T_s} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{(vi)}$$

The rate of heat conduction through the cylinder is given as follows.

$$Q = -kA\left(\frac{dT}{dr}\right)_{r=R} = -k \times (2\pi RL) \times \left(-\frac{q_g R}{2k}\right) = \pi R^2 L q_g$$

When heat conducted is dissipated to the ambient air or fluid at a temperature T_a with convective heat transfer coefficient of h, then we get:

$$\pi R^2 L q_g = h \times 2\pi RL \times (T_s - T_a)$$

$$\therefore T_s = T_a + \frac{q_g R}{2h} \quad \text{(vii)}$$

By substituting T_s in Equations (iv) and (v), we get:

$$T = T_a + \frac{q_g R}{2h} + \frac{q_g}{4k}(R^2 - r^2) \quad \text{(viii)}$$

$$T_{\max} = T_a + \frac{q_g R}{2h} + \frac{q_g R^2}{4k} \quad \text{(ix)}$$

Question 14. One meter long Nichrome heating wire with $k = 12\, W/mK$ and resistivity $= 1 \times 10^{-6}\ \Omega m$ is used in a $10\ kW$ electric heater. The heat is being dissipated to the surroundings at 293 K with heat transfer coefficient of $1000 W/m^2K$. If the maximum surface temperature of the heating element is 1300 K, determine (i) the diameter of the wire and (ii) the rate of current flow.

Solution: Let $L = 1\,m$, $k = 12\,W/mK$, $\rho = 1\times10^{-6}\ \Omega m$, $Q = 10\,kW = 10\times10^{3}\,W$, $T_a = 293\,K$, $h = 1000\,W/m^2K$ and $T_{\max} = 1300\,K$. Let D be the diameter and I be the current flowing through the wire.

(i) $$q_g = \frac{Q}{volume} = \frac{Q}{(\pi/4)D^2\times L} = \frac{10\times10^3}{(\pi/4)\times D^2\times 1} = \frac{4\times10^4}{\pi D^2}\,W/m^3$$

Since $$T_{\max} = T_a + \frac{q_g}{2h}R + \frac{q_g}{4k}R^2$$

Thus $$1300 = 293 + \frac{4\times10^4}{\pi D^2\times2\times1000}\times\left(\frac{D}{2}\right) + \frac{4\times10^4}{\pi D^2\times4\times12}\times\left(\frac{D}{2}\right)^2$$

$$\therefore D = 3.384\times10^{-3}\ m\ or\ 3.384\,mm$$ **(Ans.)**

(ii) $$R_e = \frac{\rho L}{A} = \frac{\rho L}{(\pi/4)\times D^2} = \frac{1\times10^{-6}\times1}{(\pi/4)\times(3.384\times10^{-3})^2} = 0.1112\,\Omega$$

Since $Q = I^2\times R_e$

$$\therefore I = \sqrt{\frac{Q}{R_e}} = \sqrt{\frac{10\times10^3}{0.1112}} = 299.88\,A$$ **(Ans.)**

Question 15. Derive the expressions for temperature distribution and rate of heat conduction for a solid sphere made of homogeneous and isotropic material with uniform heat generation for 1-D heat conduction (i.e., r-direction) under steady state.

Answer: The governing equation for heat conduction in one direction (r-direction) under steady state through a spherical body made of homogeneous and isotropic material with uniform heat generation can be given by below equation as,

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{q_g}{k}r^2 = 0$$

Integrating, we get: $r^2\dfrac{dT}{dr} = -\dfrac{q_g r^3}{3k} + C_1$

or $$\frac{dT}{dr} = -\frac{q_g r}{3k} + \frac{C_1}{r^2} \qquad \text{(i)}$$

Integrating again, we get:

$$T = -\frac{q_g r^2}{6k} - \frac{C_1}{r} + C_2 \qquad \text{(ii)}$$

Here, C_1 and C_2 are arbitrary constants whose values can be obtained with the help of the given boundary conditions. Equation (ii) is the general solution for temperature distribution for a sphere with uniform internal heat generation.

Consider a solid sphere of radius R having a uniform heat generation rate within its volume at a rate of q_g (W/m^3). Let T_s be the outer surface temperature and k be the constant thermal conductivity.

The boundary conditions are:

(i) $(dT/dr)=0$ at $r=0$, i.e., at the centre and

(ii) $T=T_s$ at $r=R$, i.e., at the surface.

Using first boundary condition in Equation (i), we get: $C_1=0$

Using second boundary condition and value of C_1 in equation (ii), we get:

$$T_s=-\frac{q_gR^2}{6k}+C_2 \Rightarrow \therefore C_2=T_s+\frac{q_gR^2}{6k}$$

Substituting values of C_1 and C_2 in Equation (ii), we get:

$$T=-\frac{q_gr^2}{6k}+T_s+\frac{q_gR^2}{6k}=T_s+\frac{q_g}{6k}(R^2-r^2) \qquad \text{(iii)}$$

It can be seen from Equation (iii) that the temperature distribution is parabolic and the maximum temperature will occur at the centre, i.e., at $r=0$ and its value is as follows.

$$T_{\max}=T_s+\frac{q_g}{6k}R^2 \qquad \text{(iv)}$$

Substitution of $C_1=0$ in Equation (i) gives:

$$\frac{dT}{dr}=-\frac{q_gr}{3k}$$

Heat transfer due to conduction at the outer surface of the sphere is given as follows.

$$Q=-kA\left(\frac{dT}{dr}\right)_{r=R}=-k\times(4\pi R^2)\times\left(-\frac{q_gR}{3k}\right)=\frac{4}{3}\pi R^3\times q_g$$

Above Equation shows that the heat conducted to the surface is equal to the heat generated within the sphere.

When heat conducted is dissipated to the ambient air or fluid at a temperature T_a with convective heat transfer coefficient of h, then we get the below expression.

$$\frac{4}{3}\pi R^3\times q_g=h\times 4\pi R^2\times(T_s-T_a)$$

$$\therefore T_s=T_a+\frac{q_gR}{3h}$$

By substituting this value of T_s in Equations (iii) and (iv), we get:

$$T = T_a + \frac{q_g R}{3h} + \frac{q_g}{6k}(R^2 - r^2)$$

$$T_{\max} = T_a + \frac{q_g R}{3h} + \frac{q_g}{6k} R^2$$

Question 16. During the ripening process of oranges the average heat generated is found to be $240\,W/m^2$. The average size of an orange to be 8 cm assuming as a sphere with thermal conductivity of $0.2\,W/mK$. If the outer surface of the orange is at 280 K, determine the temperature at its centre. Also evaluate the heat flow from the outer surface of the orange.

Solution: Let $Q/A = 240\,W/m^2$, $D = 8\,cm = 0.08\,m$, $k = 0.2\,W/mK$ and $T_s = 280\,K$.

$$R = D/2 = 0.08/2 = 0.04\,m$$

Since $$\frac{Q}{A} = \frac{q_g \times volume}{A} = \frac{q_g \times (4/3)\pi R^3}{4\pi R^2} = \frac{q_g R}{3}$$

$$\therefore q_g = \frac{3 \times (Q/A)}{R} = \frac{3 \times 240}{0.04} = 18000\,W/m^3$$

Using below Equation, we get:

$$T_{\max} = T_s + \frac{q_g R^2}{6k} = 280 + \frac{18000 \times 0.04^2}{6 \times 0.2} = 304\,K$$ **(Ans.)**

Since Heat conducted = Heat generated

$$\therefore Q = \frac{4}{3}\pi R^3 \times q_g = \frac{4}{3}\pi \times 0.04^3 \times 18000 = 4.825\,W$$ **(Ans.)**

Question 17. Heat generation rate in a solid sphere of radius 6 cm is $4.2 \times 10^6\ W/m^3$. The outer surface is surrounded by a fluid at 420 K having a heat transfer coefficient of $800\,W/m^2K$. If thermal conductivity of the sphere is $25\,W/mK$, determine (i) temperature at 4 cm radius and (ii) maximum temperature.

Solution: Let $R = 6\,cm = 0.06\,m$, $q_g = 4.2 \times 10^6\ W/m^3$, $T_a = 420\,K$, $h = 800\,W/m^2K$, $k = 25\,W/mK$ and $r = 4\,cm = 0.04\,m$.

(i) $$T = T_a + \frac{q_g R}{3h} + \frac{q_g}{6k}(R^2 - r^2)$$

$$\therefore T = 420 + \frac{4.2 \times 10^6 \times 0.06}{3 \times 800} + \frac{4.2 \times 10^6}{6 \times 25} \times (0.06^2 - 0.04^2) = 581\,K$$ **(Ans.)**

(ii) $$T_{\max} = T_a + \frac{q_g R}{3h} + \frac{q_g}{6k} R^2$$

$$\therefore T_{max} = 420 + \frac{4.2\times10^6\times0.06}{3\times800} + \frac{4.2\times10^6\times0.06^2}{6\times25} = 625.8\,K$$ **(Ans.)**

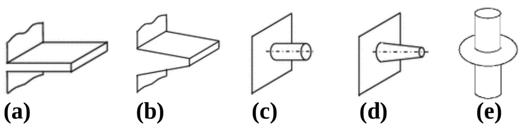
Question 18. What do you mean by fins? By design consideration name and discuss the law which governs rate of heat transfer from the fins.

Answer: The extended surfaces also called as fins are generally the thin metal strips made of highly conductive materials such as aluminium, copper, brass, etc. With the attachment of fins the effective heat transfer area on a solid surface increases thereby the rate of heat transfer may increase manifold. Fins are attached to the base material by pressing, soldering or welding. In some cases, fins may also be made integral parts of the base material by casting or extruding process.

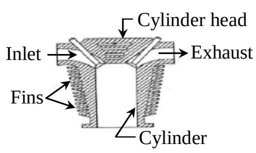
The rate of convection heat transfer from a solid surface at a temperature T_s to the surrounding fluid (gas or liquid) at a temperature T_∞ is given by the Newton's law of cooling as, $Q = hA(T_s - T_\infty)$. Here, h is the convective heat transfer coefficient and A is the surface area through which heat transfer takes place. Generally, by design considerations the temperature difference $(T_s - T_\infty)$ is fixed, therefore, the heat transfer rate can be increased either by increasing the convective heat transfer coefficient h or by increasing the surface area A. The value of h can be increased by forced convection mode using a pump, fan or a blower that may not be feasible and economical. Thus, increase in the surface area A is the only way to increase the heat transfer rate. This can be accomplished by attaching extended surfaces to the base surface.

Question 19. What are the most common types of fins? Also give some of the practical application areas of fins.

Answer: As per the requirements, fins are manufactured in different forms that may be of uniform or non-uniform (variable) cross-sections. The most common types of fins shown in below Figure are: (a) A straight fin of uniform cross-section, (b) A straight fin of non-uniform cross-section, (c) A pin fin (or spine) of uniform cross-section (d) A pin fin of non-uniform cross-section (e) An annular (or circular) fin which circumferentially attaches to a cylinder.



Some of the practical application areas of fins are: (i) Air cooling of IC engines such as in scooters and motorcycles (refer below Figure), (ii) Cooling of air compressors, electric motors and transformers, (iii) Cooling of electronic equipments, (iv) Radiators for automobiles, heat exchangers and condensing coil of refrigerators.

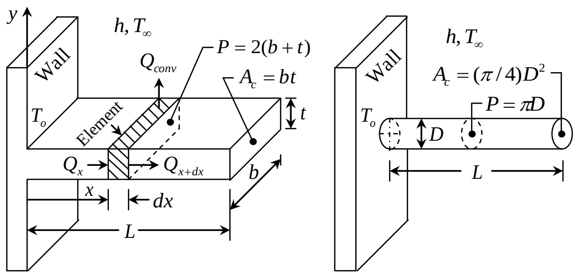


The temperature distribution in a fin can be obtained by considering the fin as a metallic plate attached at its base to a heated wall and transferring heat to a fluid by convection. It depends on the properties of its material and the surrounding fluid. The heat flow through a fin is by conduction. Generally, a fin is thin, therefore, heat conduction through it can be considered one-dimensional.

Question 19. Derive an expression for the governing differential equation for the fin of uniform cross-section. Also give the assumptions made during heat conduction analysis through a fin.

Answer: The various assumptions made during heat conduction analysis through a fin are: (i) Heat conduction through the fin is one-dimensional and steady, (ii) No heat generation within the fin, (iii) Thermal conductivity (k) of the fin material is constant, (iv) The convective heat transfer coefficient (h) over the entire fin surface is uniform, (v) There is no contact thermal resistance between the fin and base material, and (vi) Radiation heat transfer is negligible.

Below Figure shows fins of uniform cross-sections protruding from a wall surface. Figure (a) shows a rectangular fin and Figure (b) shows a pin fin of uniform cross-section. Let L be the length, b be the width and t be the thickness of the rectangular fin. The other characteristic dimensions of the fins are its constant cross-sectional area $A_c = bt$ and perimeter $P = 2(b+t)$. Let T_o be the base temperature of fin, T_∞ be the temperature of surrounding fluid and h be the convective heat transfer coefficient. The cross-sectional area and perimeter of a pin fin of diameter D are $A_c = (\pi/4)D^2$ and $P = \pi D$, respectively.



(a) Rectangular fin (b) Pin fin

For determining the governing differential equation for the fin, applying energy balance to an infinitesimal element of the fin of length dx at a distance x from the base surface as shown in Figure (a).

Heat conducted in the element at x is given by Fourier's law as below.

$$Q_x = -kA_c \frac{dT}{dx}$$

Heat conducted out from the element at $x + dx$ is given as below.

$$Q_{x+dx} = Q_x + \frac{d}{dx}(Q_x)dx = -kA_c \frac{dT}{dx} - kA_c \frac{d^2T}{dx^2} dx$$

Heat convected from the surface of the element to the surrounding is given by Newton's law of cooling as below.

$$Q_{conv} = hA(T - T_\infty) = h(Pdx)(T - T_\infty)$$

Here, the temperature T of the fin is assumed uniform for the infinitesimal element.

Thus energy balance for the element gives:

$$Q_x = Q_{x+dx} + Q_{conv}$$

$$-kA_c\frac{dT}{dx} = -kA_c\frac{dT}{dx} - kA_c\frac{d^2T}{dx^2}dx + h(Pdx)(T - T_\infty)$$

$$kA_c\frac{d^2T}{dx^2}dx - h(Pdx)(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0 \qquad \text{(i)}$$

Here $m^2 = \frac{hP}{kA_c}$ or $m = \sqrt{\frac{hP}{kA_c}}$ (ii)

Now defining the excess temperature θ as follows,

$$\theta = T - T_\infty$$

Since surrounding temperature T_∞ is constant, thus by differentiation, we get:

$$\frac{d\theta}{dx} = \frac{dT}{dx} \text{ and } \frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

Thus Equation (i) becomes,

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \text{(iii)}$$

Equation (iii) is the governing differential equation for the fin of uniform cross-section which describes the temperature as a function of x and m. The general solution of this linear homogeneous, second order ordinary differential equation can be given in the following form.

$$\theta = C_1e^{mx} + C_2e^{-mx} \qquad \text{(iv)}$$

The constants C_1 and C_2 can be determined by using two relevant boundary conditions.

Question 21. Derive expressions for temperature distribution and rate of heat transfer for infinitely long fin.

Answer: The governing differential equation for the fin of uniform cross-section which describes the temperature as a function of x and m is given as,

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0\text{, where, } m^2 = \frac{hP}{kA_c} \text{ or } m = \sqrt{\frac{hP}{kA_c}}$$

The general solution of this linear homogeneous, second order ordinary differential equation can be given in the following form.

$$\theta = C_1e^{mx} + C_2e^{-mx} \qquad \text{(i)}$$

When fin is very long, the temperature at the tip of the fin (i.e., at $x = L$) becomes equal to that of the surrounding fluid. The constants C_1 and C_2 can be determined by using below boundary conditions.

(i) $T = T_o$ at $x = 0$; $T - T_\infty = T_o - T_\infty$ at $x = 0$; $\theta = \theta_o$ at $x = 0$

(ii) $T = T_\infty$ at $x = L = \infty$; $T - T_\infty = T_\infty - T_\infty$ at $x = \infty$; $\theta = 0$ at $x = \infty$

Substitution of first boundary condition in Equation (i) gives:

$$\theta_o = C_1 + C_2 \qquad \text{(ii)}$$

Using second b. c. in above Equation gives: $0 = C_1 e^{m\infty} + C_2 e^{-m\infty}$

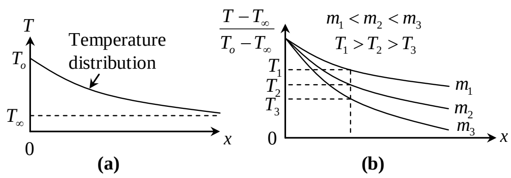
Since $C_2 e^{-m\infty} = 0$, the equality is valid only when $C_1 = 0$, $\therefore C_1 = 0$

From Equations (ii), we get: $C_2 = \theta_o$

Substituting the values of C_1 and C_2 in Equation (ii), we get:

$$\theta = \theta_o e^{-mx} \text{ or } \frac{\theta}{\theta_o} = \frac{T - T_\infty}{T_o - T_\infty} = e^{-mx} \qquad \text{(iii)}$$

Equation (iii) gives the temperature distribution in an infinitely long fin of uniform cross-section along its length. This exponential temperature distribution is shown in Figure (a) from which it can be seen that the temperature drops rapidly near the base of the fin and progressively reaches the ambient temperature at some length. It means most of the heat is dissipated near the base of the fin and the area near the fin tip makes little or no contribution to heat transfer to the extent as the lateral area near the base of the fin. Therefore, increase in length of the fin beyond a certain limit results in excessive weight, wastage of material and increased size and thus increased cost with no benefit in return. A tapered fin will be a better choice since its lateral surface area is more near the base which results in increased heat dissipation rate. The dependence of dimensionless temperature given by Equation (iii) along the length of the fin for different values of parameter m is shown in Figure (b). This plot shows that temperature is inversely proportional to the values of parameter m, i.e., higher the value of m lower be the temperature at a distance x from the base of the fin.



Convective heat loss from the entire fin is given as follows.

$$Q = \int_0^\infty hPdx(T - T_\infty)$$

But $\quad T - T_\infty = (T_o - T_\infty)e^{-mx} \qquad$ [From Equation (iii)]

Thus $Q = \int_0^\infty hPdx(T_o - T_\infty)e^{-mx} = hP(T_o - T_\infty)\int_0^\infty e^{-mx}dx$

$$Q = hP(T_o - T_\infty) \times \frac{1}{m} = hP(T_o - T_\infty) \times \sqrt{\frac{kA_c}{hP}}$$

$$\therefore Q = \sqrt{hPkA_c}(T_o - T_\infty) = \sqrt{hPkA_c} \times \theta_o \qquad \text{(iv)}$$

Above Equation gives the rate of heat transfer through the fin.

Alternatively: From Equation (iii), we get:

$$T = (T_o - T_\infty)e^{-mx} + T_\infty$$

$$\left(\frac{dT}{dx}\right) = -m(T_o - T_\infty)e^{-mx} \qquad \text{(Differentiating)}$$

The rate of heat flow across the base of the fin is given by Fourier's law as follows.

$$Q = -kA_c\left(\frac{dT}{dx}\right)_{x=0} = -kA_c\left[-m(T_o - T_\infty)e^{-mx}\right]_{x=0} = kA_c m(T_o - T_\infty)$$

Substituting value of $m = \sqrt{hP / kA_c}$, we get:

$$Q = kA_c \times \sqrt{\frac{hP}{kA_c}} \times (T_o - T_\infty) = \sqrt{hPkA_c}(T_o - T_\infty) = \sqrt{hPkA_c} \times \theta_o$$

The above expression for rate of heat conduction is same as Equation (iv).

Question 21. Derive expressions for temperature distribution and rate of heat transfer for a finite long fin with insulated tip.

Answer: The governing differential equation for the fin of uniform cross-section which describes the temperature as a function of x and m is given as,

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \text{, where, } m^2 = \frac{hP}{kA_c} \text{ or } m = \sqrt{\frac{hP}{kA_c}}$$

The general solution of this linear homogeneous, second order ordinary differential equation can be given in the following form.

$$\theta = C_1e^{mx} + C_2e^{-mx} \qquad \text{(i)}$$

$$\frac{d\theta}{dx} = mC_1e^{mx} - mC_2e^{-mx} \qquad \text{(ii)}$$

The constants C_1 and C_2 can be determined by using below boundary conditions.

(i) $\theta = T_o - T_\infty = \theta_o$ at $x = 0$; (ii) $\frac{d\theta}{dx} = 0$ at $x = L$

Applying 1st b.c. to Eq. (i), we get: $\theta_o = C_1 + C_2$ (iii)

Applying 2nd b.c. to Eq. (ii), we get: $0 = mC_1e^{mL} - C_2e^{-mL} \Rightarrow \therefore C_2 = C_1e^{2mL}$

Substituting this value of $C_2 = C_1e^{2mL}$ in equation (iii), we get:

$$\theta_o = C_1 + C_1e^{2mL} = C_1(1+e^{2mL}) \Rightarrow C_1 = \frac{\theta_o}{(1+e^{2mL})}$$

Thus $C_2 = \dfrac{\theta_o}{(1+e^{2mL})}e^{2mL} = \dfrac{\theta_o}{e^{-2mL}+1}$

Substituting the values of C_1 and C_2 in equation (i), we get:

$$\theta = \frac{\theta_o}{1+e^{2mL}}e^{mx} + \frac{\theta_o}{e^{-2mL}+1}e^{-mx} = \theta_o\left[\frac{e^{mx}}{1+e^{2mL}} + \frac{e^{-mx}}{e^{-2mL}+1}\right]$$

$$\frac{\theta}{\theta_o} = \left[\frac{e^{-mL}}{e^{-mL}} \times \frac{e^{mx}}{(1+e^{2mL})} + \frac{e^{mL}}{e^{mL}} \times \frac{e^{-mx}}{(e^{-2mL}+1)}\right] = \left[\frac{e^{m(x-L)}}{e^{-mL}+e^{mL}} + \frac{e^{m(L-x)}}{e^{-mL}+e^{mL}}\right]$$

$$\frac{\theta}{\theta_o} = \left[\frac{e^{m(x-L)}+e^{m(L-x)}}{e^{mL}+e^{-mL}}\right] = \left[\frac{e^{m(L-x)}+e^{-m(L-x)}}{e^{mL}+e^{-mL}}\right]$$

Since $\cosh m(L-x) = \dfrac{e^{m(L-x)}+e^{-m(L-x)}}{2}$ and $\cosh mL = \dfrac{e^{mL}+e^{-mL}}{2}$

$$\therefore \frac{\theta}{\theta_o} = \frac{T-T_\infty}{T_o-T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

Now $T = T_\infty + (T_o - T_\infty)\dfrac{\cosh m(L-x)}{\cosh mL}$

Thus $\dfrac{dT}{dx} = (T_o - T_\infty)\dfrac{\sinh m(L-x)}{\cosh mL}(-m)$

$$\left(\frac{dT}{dx}\right)_{x=0} = -m(T_o - T_\infty)\tanh mL$$

The rate of heat flow from the fin is given as,

$$Q = -kA_c\left(\frac{dT}{dx}\right)_{x=0} = -kA_c \times \left[-m(T_o - T_\infty)\tanh mL\right]$$

$$Q = kA_c m(T_o - T_\infty)\tanh mL$$

Substituting value of $m = \sqrt{hP/kA_c}$, we get:

$$Q = \sqrt{hPkA_c}(T_o - T_\infty)\tanh mL$$

Question 22. Derive expressions for temperature distribution and rate of heat transfer for a finite long fin dissipating heat at its tip by convection.

Answer: The governing differential equation for the fin of uniform cross-section which describes the temperature as a function of x and m is given as,

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \text{, where, } m^2 = \frac{hP}{kA_c} \text{ or } m = \sqrt{\frac{hP}{kA_c}}$$

The general solution of this linear homogeneous, second order ordinary differential equation can be given in the following form.

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \qquad \text{(i)}$$

$$\frac{d\theta}{dx} = mC_1 e^{mx} - mC_2 e^{-mx} \qquad \text{(ii)}$$

The constants C_1 and C_2 can be determined by using below boundary conditions.

(i) $\theta = T_o - T_\infty = \theta_o$ at $x = 0$

(ii) The fin is losing heat at the tip, i.e., heat conducted at the fin at $x = L$ equals heat convected from the end to the surroundings. Also at the tip of the fin $A_c = A_s = A$. Therefore, $-kA\left(\frac{dT}{dx}\right)_{x=L} = hA(T - T_\infty) \Rightarrow \frac{dT}{dx} = -\frac{h\theta}{k}$ at $x = L$

or $$\frac{d\theta}{dx} = -\frac{h\theta}{k} \text{ at } x = L \qquad \text{(iii)} \qquad \left[\because \frac{d\theta}{dx} = \frac{dT}{dx}\right]$$

Using first b.c. in equation (i), we get: $\theta_o = C_1 + C_2 \Rightarrow C_2 = \theta_o - C_1$

From 1st b.c., 2nd b.c. and expression (iii), at $x = L$, we have,

$$mC_1 e^{mL} - mC_2 e^{-mL} = -\frac{h}{k}\left[C_1 e^{mL} + C_2 e^{-mL}\right]$$

$$C_1 e^{mL} - (\theta_o - C_1)e^{-mL} = -\frac{h}{km}\left[C_1 e^{mL} + (\theta_o - C_1)e^{-mL}\right] \qquad [\because C_2 = \theta_o - C_1]$$

$$C_1 e^{mL} - \theta_o e^{-mL} + C_1 e^{-mL} = -\frac{h}{km}C_1 e^{mL} - \frac{h}{km}\theta_o e^{-mL} + \frac{h}{km}C_1 e^{-mL}$$

$$C_1\left[(e^{mL} + e^{-mL}) + \frac{h}{km}(e^{mL} - e^{-mL})\right] = \theta_o e^{-mL}\left[1 - \frac{h}{km}\right]$$

$$\therefore C_1 = \frac{\theta_o[1 - (h/km)]e^{-mL}}{[(e^{mL} + e^{-mL}) + (h/km)(e^{mL} - e^{-mL})]}$$

and $$C_2 = \theta_o - \frac{\theta_o[1 - (h/km)]e^{-mL}}{[(e^{mL} + e^{-mL}) + (h/km)(e^{mL} - e^{-mL})]} \qquad [\because C_2 = \theta_o - C_1]$$

$$C_2 = \theta_o\left[1 - \frac{[1-(h/km)]e^{-mL}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}\right]$$

$$C_2 = \theta_o\left[\frac{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})-e^{-mL}+(h/km)e^{-mL}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}\right]$$

$$C_2 = \theta_o\left[\frac{e^{mL}+e^{-mL}+(h/km)e^{mL}-(h/km)e^{-mL}-e^{-mL}+(h/km)e^{-mL}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}\right]$$

$$\therefore C_2 = \frac{\theta_o[1+(h/km)]e^{mL}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}$$

Now substituting the values of C_1 and C_2 in expression (i), we get:

$$\theta = \frac{\theta_o[1-(h/km)]e^{-mL}}{[(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})]}e^{mx} + \frac{\theta_o[1+(h/km)]e^{mL}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}e^{-mx}$$

$$\frac{\theta}{\theta_o} = \frac{e^{-mL+mx}-(h/km)e^{-mL+mx}+e^{mL-mx}+(h/km)e^{mL-mx}}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}$$

$$\frac{\theta}{\theta_o} = \frac{[e^{m(L-x)}+e^{-m(L-x)}]+(h/km)[e^{m(L-x)}-e^{-m(L-x)}]}{(e^{mL}+e^{-mL})+(h/km)(e^{mL}-e^{-mL})}$$

$$\therefore \frac{\theta}{\theta_o} = \frac{T-T_\infty}{T_o-T_\infty} = \frac{\cosh[m(L-x)]+(h/km)\sinh[m(L-x)]}{\cosh(mL)+(h/km)\sinh(mL)}$$

or $$\theta = \theta_o\frac{\cosh[m(L-x)]+(h/km)\sinh[m(L-x)]}{\cosh(mL)+(h/km)\sinh(mL)}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \theta_o\frac{-m\sinh(mL)-(h/km)\cosh(mL)}{\cosh(mL)+(h/km)\sinh(mL)}$$

The rate of heat flow from the fin is given as,

$$Q = -kA_c\left(\frac{dT}{dx}\right)_{x=0} = -kA_c\times\theta_o\frac{-m\sinh(mL)-(h/km)m\cosh(mL)}{\cosh(mL)+(h/km)\sinh(mL)}$$

$$Q = kA_c m\times\theta_o\frac{\sinh(mL)+(h/km)\cosh(mL)}{\cosh(mL)+(h/km)\sinh(mL)}$$

Substituting value of $m = \sqrt{hP/kA_c}$ and rearranging, we get:

$$Q = \sqrt{hPkA_c}\times\theta_o\frac{\tanh(mL)+(h/km)\tanh(mL)}{1+(h/km)\tanh(mL)}$$

$$\therefore Q = \sqrt{hPkA_c} \times (T_o - T_\infty) \frac{\tanh(mL) + (h/km)\tanh(mL)}{1 + (h/km)\tanh(mL)}$$

CHAPTER - 7

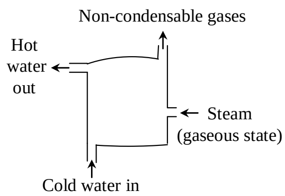
HEAT EXCHANGER

Question 7.1. Define a heat exchanger. Give its examples and applications.

Answer: A thermal device used to exchange of heat between two fluids at different temperatures is called a heat exchanger, e.g., boiler, condenser and radiator. Heat exchangers find applications in refrigerating and air-conditioning systems, power plants, food processing systems, chemical reactors and aeronautical (or space) devices.

Question 7.2. What do you mean by direct and indirect contact heat exchangers? Briefly explain with examples.

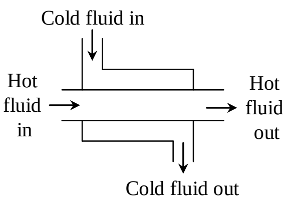
(i) Direct contact heat exchangers: A direct contact heat exchanger is a device in which hot and cold fluids come in direct contact thereby heat transfer occurs. Here, fluids will be in different states such as one fluid is in gaseous state (steam) and other is in liquid state, e.g., cooling towers, jet condensers and open feed water heaters.

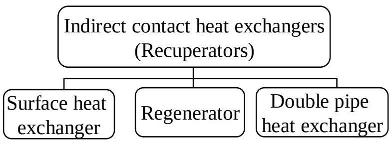


Direct contact heat exchanger

(ii) Indirect contact heat exchangers: An indirect contact heat exchanger is a device in which hot and cold fluids do not come in direct contact and are separated by a wall through which heat transfer takes place. Such heat exchangers are also called as recuperators. The separating wall may be a simple plane wall or a tube or a configuration involving fins, baffles and multiple passes of tubes. The indirect contact heat exchangers also include surface heat exchangers and regenerators.

When the hot and cold fluids are separated by double pipe arrangement (or concentric tube) it is called double pipe heat exchanger or concentric tube heat exchanger.



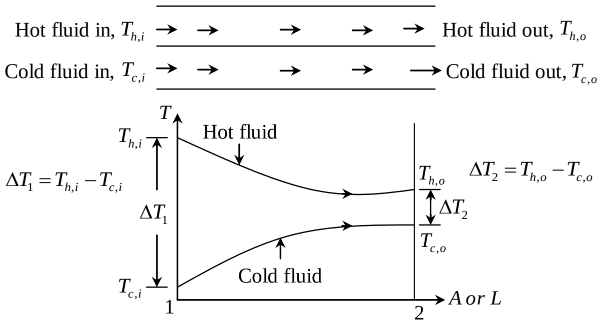


In regenerators same space is used alternately by hot and cold fluids between which heat is exchanged. It generally operates periodically. Heat absorbed during the flow of hot fluid by the walls of the heat exchanger is transferred to the cold fluid when it is made to flow through the heat exchanger after the flow of hot fluid. A regenerator finds application in preheaters for steam power plants.

Question 7.3. Give the classification of heat exchanger as per flow arrangements of fluids.

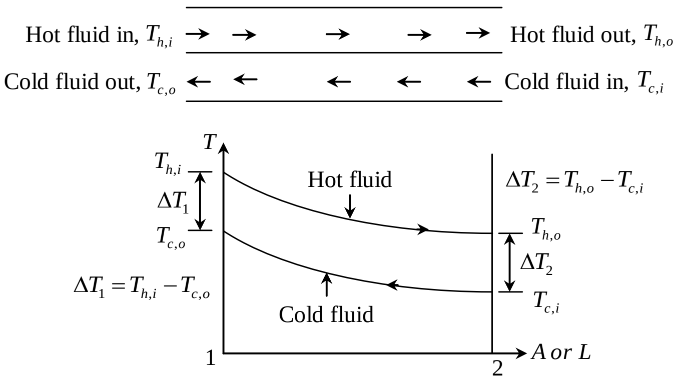
Answer: According to flow arrangements of fluids (or direction of fluids) heat exchangers are given below.

(i) **Parallel flow heat exchanger**: In a parallel flow heat exchanger, hot and cold fluids flow in the same direction. The temperature difference (ΔT) of the fluids is maximum at its inlet and minimum at the outlet and it keep on decreasing in the flow direction.



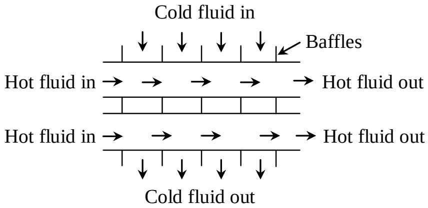
Parallel flow heat exchanger

(ii) **Counter flow heat exchanger**: In a counter flow heat exchanger, hot and cold fluids flow in the opposite directions. The inlet of hot fluid and outlet of cold fluid is at one end while the outlet of hot fluid and inlet of cold fluid is at other end of the heat exchanger. The temperature difference (ΔT) of the fluids remains almost constant for the whole length (L) of the heat exchanger.



Counter flow heat exchanger

(iii) **Cross-flow heat exchanger**: In a cross-flow heat exchanger the hot and cold fluids generally flow at right angle to each other.



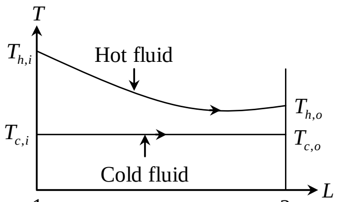
Cross-flow heat exchanger

Question 7.4. Define evaporator and condenser.

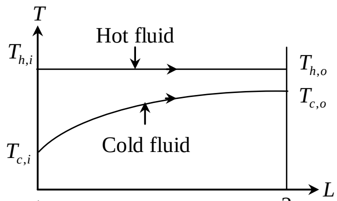
Answer: Depending on the physical state of the fluids the heat exchangers are namely evaporator and condenser. One of the fluids flowing through these heat exchangers changes phase.

(i) **Evaporator**: In an evaporator, cold fluid evaporates at constant temperature (provided pressure remains constant) while the temperature of hot fluid decreases from inlet to outlet.

(i) **Condenser**: In a condenser, hot fluid condenses at constant temperature (provided pressure remains constant) by transferring the heat (latent heat) to the cold fluid. The temperature of cold fluid increases from inlet to outlet.



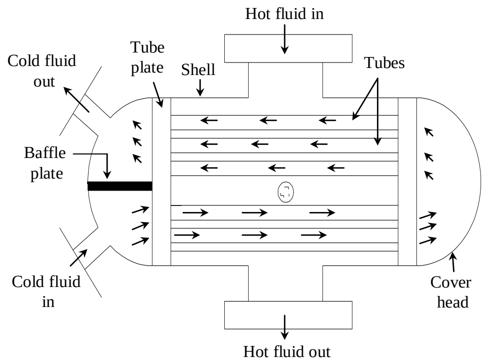
Evaporator



Condenser

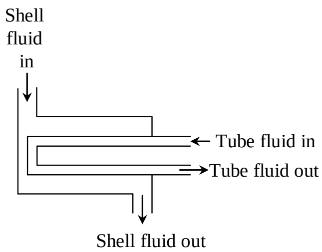
Question 7.5. Briefly explain shell and tube heat exchanger and its types.

Answer: In a shell and tube heat exchanger, one of the fluids is passed through a bundle of tubes enclosed by a shell. The other fluid flows through the shell which flows over the outside surface of tubes. Here, clod fluid passes through the tubes (also called tube fluid) and hot fluid flows through the shell (also called shell fluid). Baffles are used to guide the flow of fluid in the shell and it also generates turbulence in the flow which causes better heat transfer due to increased heat transfer coefficient. These heat exchangers are commonly used because they can be constructed with large heat transfer surfaces in relatively small volume. These are suitable for heating, cooling, evaporating or condensing applications

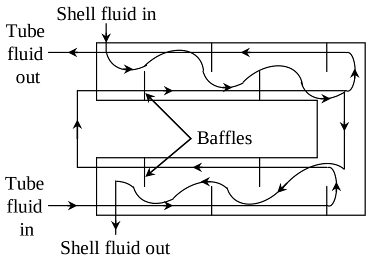


Shell and tube heat exchanger

When the two fluids flow through the heat exchanger only once, it is called single pass but when both fluids passes through the heat exchanger more than once it is called multi-pass heat exchanger. One shell and two tube pass and two-shell four tube pass are some of the multi-pass flow heat exchangers.



One shell two tube-pass heat exchanger



Two shell four tube-pass heat exchanger

Question 7.6. List the variables involved in heat transfer analysis? What do you mean by logarithmic mean temperature difference (LMTD).

Answer: The main variables involved in the thermal analysis of a heat exchanger are inlet fluid temperature (T_i), outlet fluid temperature (T_o), the overall heat transfer coefficient (U), total surface area for heat transfer (A) and the total heat transfer rate (Q). Let subscripts h denotes the hot fluid, c denotes the cold fluid, i denotes the fluid at inlet, and o denotes the fluid at outlet of the heat exchanger. The hot fluid transfers a part of its energy to the cold fluid. Therefore, enthalpy of the cold fluid increases and there will be a corresponding decrease in enthalpy of the hot fluid as given below.

$$Q = \dot{m}_h c_h (T_{h,i} - T_{h,o}) = C_h (T_{h,i} - T_{h,o})$$

$$Q = \dot{m}_c c_c (T_{c,o} - T_{c,i}) = C_c (T_{c,o} - T_{c,i})$$

Here, $\dot{m}_h$ = mass flow rate of hot fluid in kg/s, $\dot{m}_c$ = mass flow rate of cold fluid in kg/s, c_h = specific heat of the hot fluid in $J/kg.K$, c_c = specific heat of the cold fluid in $J/kg.K$, $C_h = \dot{m}_h c_h$ is the heat capacity (or heat capacity rate) of hot fluid and $C_c = \dot{m}_c c_c$ is the heat capacity (or heat capacity rate) of cold fluid.

The temperature difference between the hot and cold fluids $\Delta T = T_h - T_c$ varies with position in the heat exchanger, therefore, the actual rate of heat transfer will be given by the expression, $Q = UA\Delta T_m$. Here, ΔT_m is the appropriate mean temperature difference across the heat exchanger structure.

For parallel and counter flow heat exchangers this mean temperature difference will be having logarithmic relation, and can be given by ΔT_{lm} (log mean temperature difference or LMTD). Therefore, the total heat transfer rate between the hot and cold fluids can be calculated by using the below equation.

$$Q = UA\Delta T_{lm} = UA \times (LMTD)$$

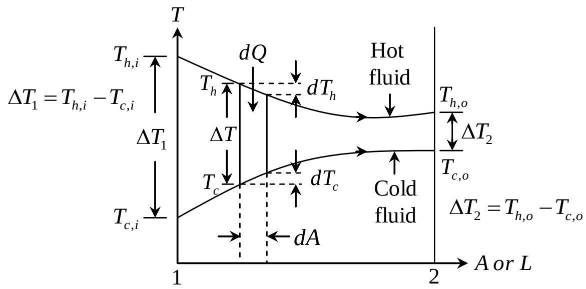
Question 7.7. Give the assumptions for the analysis of LMTD for parallel and counter flow heat exchangers.

Answer: For parallel and counter flow heat exchangers the logarithmic mean temperature difference (LMTD) analyses are described by considering the below assumptions.

(i) The flow conditions are steady, (ii) The overall heat transfer coefficient (U) is constant throughout the heat exchanger, (iii) The specific heats of both the fluids are constant, (iv) The potential and kinetic energies changes are negligible, (v) The heat exchanger is perfectly insulated, i.e., no heat loss to the surroundings, (vi) Axial conduction along the tube is negligible and (vii) There is no change of phase either of the fluid during the heat transfer.

Question 7.8. Derive an expression for LMTD for a parallel flow heat exchanger.

Answer: The LMTD method is preferred in evaluating the performance of a heat exchanger when the inlet and outlet temperatures of the fluids are either known or can be easily determined. Figure shows variation of temperatures of the hot and cold fluid streams in a parallel flow double pipe heat exchanger along its length.



Parallel flow heat exchanger

The heat transfer between the cold and hot fluids for an elementary area dA is given by,

$$dQ = UdA(T_h - T_c) = UdA\Delta T$$

The temperature of hot fluid decreases and that of cold fluid increases along the length of heat exchanger (L). The hot fluid is cooled by dT_h and the cold fluid is heated by dT_c, therefore, negative and positive signs are shown in below equations.

The energy balance to differential element between hot and cold fluids gives,

$$dQ = -\dot{m}_h c_h dT_h = -C_h dT_h \Rightarrow dT_h = -\frac{dQ}{C_h}$$

$$dQ = +\dot{m}_c c_c dT_c = +C_c dT_c \Rightarrow dT_c = \frac{dQ}{C_c}$$

Since, $\Delta T = T_h - T_c$

Thus, $d(\Delta T) = dT_h - dT_c$ (differentiating)

Substituting values of dT_h and dT_c, we get,

$$d(\Delta T) = -\frac{dQ}{C_h} - \frac{dQ}{C_c} = -dQ\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Substituting $dQ = UdA\Delta T$ in above equation, we get,

$$d(\Delta T) = -UdA\Delta T\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Or $$\frac{d(\Delta T)}{\Delta T} = -U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)dA$$

Integrating the above expression from the inlet section 1 to exit section 2, we get,

$$\int_{\Delta T_1}^{\Delta T_2} \frac{d(\Delta T)}{\Delta T} = -U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\int_1^2 dA$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

We know that,

$$Q = \dot{m}_h c_h (T_{h,i} - T_{h,o}) = C_h (T_{h,i} - T_{h,o}) \Rightarrow C_h = \frac{Q}{(T_{h,i} - T_{h,o})}$$

And $$Q = \dot{m}_c c_c (T_{c,o} - T_{c,i}) = C_c (T_{c,o} - T_{c,i}) \Rightarrow C_c = \frac{Q}{(T_{c,o} - T_{c,i})}$$

Substituting the values of C_h and C_c in above equation, we get,

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}[(T_{h,i} - T_{h,o}) + (T_{c,o} - T_{c,i})]$$

Or $$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}[(T_{h,i} - T_{c,i}) - (T_{h,o} - T_{c,o})]$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}(\Delta T_1 - \Delta T_2)$$

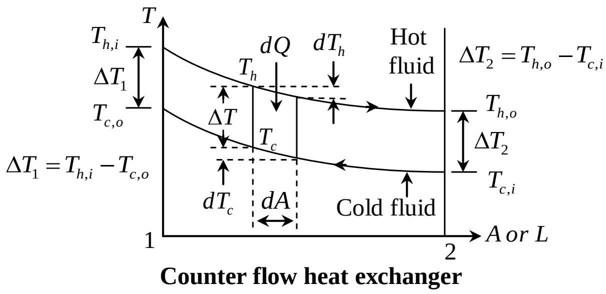
Or $$Q = UA\frac{(\Delta T_2 - \Delta T_1)}{\ln(\Delta T_2 / \Delta T_1)} = UA\Delta T_{lm} \qquad [\because Q = UA\Delta T_{lm}]$$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_2 - \Delta T_1)}{\ln(\Delta T_2 / \Delta T_1)} = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1 / \Delta T_2)}$$

Here $\Delta T_1 = T_{h,i} - T_{c,i}$ and $\Delta T_2 = T_{h,o} - T_{c,o}$

Question 7.9. Derive an expression for LMTD for a counter flow heat exchanger.

Answer: Below Figure shows the variation of temperatures of the hot and cold fluid streams in a counter flow double pipe heat exchanger along its length. Here, hot and cold fluids flow in opposite direction, therefore the outlet temperature of cold fluid may exceed the outlet temperature of hot fluid.



Counter flow heat exchanger

The temperature of hot and cold fluids decrease along the length of heat exchanger (L). Applying the energy balance to differential element, we get:

$$dQ = -\dot{m}_h c_h dT_h = -C_h dT_h \Rightarrow dT_h = -\frac{dQ}{C_h}$$

$$dQ = -\dot{m}_c c_c dT_c = -C_c dT_c \Rightarrow dT_c = -\frac{dQ}{C_c}$$

$$\because \Delta T = T_h - T_c$$

$$\therefore d(\Delta T) = dT_h - dT_c$$

Substituting values of dT_h and dT_c, we get,

$$d(\Delta T) = -\frac{dQ}{C_h} + \frac{dQ}{C_c} = -dQ\left(\frac{1}{C_h} - \frac{1}{C_c}\right)$$

The heat transfer between the cold and hot fluids for an elementary area dA is given as $dQ = UdA(T_h - T_c) = UdA\Delta T$. Substituting this value of dQ in above equation, we have,

$$d(\Delta T) = -UdA\Delta T\left(\frac{1}{C_h} - \frac{1}{C_c}\right)$$

Or $$\frac{d(\Delta T)}{\Delta T} = -U\left(\frac{1}{C_h} - \frac{1}{C_c}\right)dA$$

Integrating the above expression from the inlet section 1 to exit section 2,

$$\int_{\Delta T_1}^{\Delta T_2} \frac{d(\Delta T)}{\Delta T} = -U\left(\frac{1}{C_h} - \frac{1}{C_c}\right)\int_1^2 dA$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -UA\left(\frac{1}{C_h} - \frac{1}{C_c}\right)$$

We know that,

$$C_h = \frac{Q}{(T_{h,i} - T_{h,o})} \qquad [\because Q = C_h(T_{h,i} - T_{h,o})]$$

And $$C_c = \frac{Q}{(T_{c,o} - T_{c,i})} \qquad [\because Q = C_c(T_{c,o} - T_{c,i})]$$

Substituting the values of C_h and C_c in above equation,

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}[(T_{h,i} - T_{h,o}) - (T_{c,o} - T_{c,i})]$$

Or $$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}[(T_{h,i} - T_{c,o}) - (T_{h,o} - T_{c,i})]$$

$$\ln\left(\frac{\Delta T_2}{\Delta T_1}\right) = -\frac{UA}{Q}(\Delta T_1 - \Delta T_2)$$

Or $Q = UA\frac{(\Delta T_2 - \Delta T_1)}{\ln(\Delta T_2 / \Delta T_1)} = UA\Delta T_{lm}$ $[\because Q = UA\Delta T_{lm}]$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_2 - \Delta T_1)}{\ln(\Delta T_2 / \Delta T_1)} = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1 / \Delta T_2)}$$

Here, $\Delta T_1 = T_{h,i} - T_{c,o}$ and $\Delta T_2 = T_{h,o} - T_{c,i}$

The LMTD for a counter flow heat exchanger is always greater than that for a parallel flow heat exchanger. Therefore, counter flow heat exchanger transfers more heat than a parallel flow heat exchanger. In other words, a counter flow heat exchanger needs a smaller heating surface area for the same rate of heat transfer, hence it is generally preferred.

Note: In a case when $\Delta T_1 = \Delta T_2$, then LMTD from above equation become indeterminate. Therefore, in such a situation, LMTD becomes average of the inlet and outlet temperature difference.

That is, $\Delta T_{lm} = LMTD = \frac{(\Delta T_1 + \Delta T_2)}{2}$

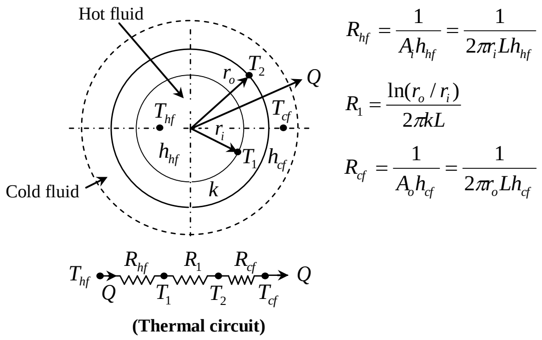
Question 7.10. Derive the overall heat transfer coefficient for a double pipe heat exchanger. Also define fouling and its effect on overall heat transfer coefficient.

Answer: Here, various thermal resistances are given as below.

$$R_{hf} = \frac{1}{A_i h_{hf}} = \frac{1}{2\pi r_i L h_{hf}}\ ;\ R_1 = \frac{\ln(r_o / r_i)}{2\pi k L}\ ;\ R_{cf} = \frac{1}{A_o h_{cf}} = \frac{1}{2\pi r_o L h_{cf}}$$

Under steady state conditions, the rate of heat transfer from the hot to cold fluid through each layer of the exchanger remains constant and it can be given as,

$$Q = \frac{T_{hf} - T_1}{R_{hf}} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_{cf}}{R_{cf}}$$



(Thermal circuit)

Thus $T_{hf} - T_1 = QR_{hf}\ ;\ T_1 - T_2 = QR_1\ ;\ T_2 - T_{cf} = QR_{cf}$

Adding these expressions, we get: $T_{hf} - T_{cf} = Q[R_{hf} + R_1 + R_{cf}]$

$$\therefore Q = \frac{T_{hf} - T_{cf}}{R_{hf} + R_1 + R_{cf}} = \frac{(T_{hf} - T_{cf})}{\left[\frac{1}{2\pi r_i L h_{hf}} + \frac{\ln(r_o / r_i)}{2\pi k L} + \frac{1}{2\pi r_o L h_{cf}}\right]}$$

The rate of heat transfer from the hot to cold fluid can be written as below.

$$Q = UA(T_{hf} - T_{cf})$$

Here, U is the overall heat transfer coefficient and A is the area normal to the direction of heat flow. For a hollow cylinder the area varies with radius. Therefore, it becomes necessary to specify the area on which U is based. U_i is the overall heat transfer coefficient based on inner area and U_o is the overall heat transfer coefficient based on outer area.

Thus $Q = U_i A_i (T_{hf} - T_{cf}) = U_o A_o (T_{hf} - T_{cf})$

or $Q = U_i \times 2\pi r_i L \times (T_{hf} - T_{cf}) = U_o \times 2\pi r_o L \times (T_{hf} - T_{cf})$

The heat transfer rate is given by,

$$Q = U_i \times 2\pi r_i L = \frac{1}{\left[\frac{1}{2\pi r_i h_{hf} L} + \frac{\ln(r_o / r_i)}{2\pi k L} + \frac{1}{2\pi r_o h_{cf} L}\right]}$$

$$\therefore U_i = \frac{1}{\left[\frac{1}{h_{hf}} + \frac{r_i}{k} \ln\left(\frac{r_o}{r_i}\right) + \frac{r_i}{r_o} \frac{1}{h_{cf}}\right]}$$

The heat transfer rate is given by,

$$Q = U_o \times 2\pi r_o L = \frac{1}{\left[\frac{1}{2\pi r_i h_{hf} L} + \frac{\ln(r_o / r_i)}{2\pi k L} + \frac{1}{2\pi r_o h_{cf} L}\right]}$$

$$\therefore U_o = \frac{1}{\left[\frac{r_o}{r_i} \frac{1}{h_{hf}} + \frac{r_o}{k} \ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_{cf}}\right]}$$

Question 7.11. Define fouling and fouling factor. Also give the expressions for finding the overall heat transfer coefficient when fouling is considered for a double pipe heat exchanger.

Answer: The deposit (rust, scale, slit or cake) formed on the heat transfer surface of the heat exchangers is called fouling. Fouling increases the thermal resistance, thus decreases the heat transfer rate. Fouling factors F is introduced to include the effect of fouling. F_i and F_o are the fouling factors based on inner and outer areas, respectively.

Fouling factors are determined experimentally by testing the heat exchanger in both the clean and dirty conditions as,

$$F = \frac{1}{U_{dirty}} - \frac{1}{U_{clean}}$$

Scale resistance due to fouling is given by,

$$\text{Scale Resistance} = \frac{F_i}{A_i} + \frac{F_o}{A_o} = \frac{F_i}{2\pi r_i L} + \frac{F_o}{2\pi r_o L}$$

The heat transfer rate is given by,

$$Q = U_i \times 2\pi r_i L = \frac{1}{\left[\frac{1}{2\pi r_i h_{hf} L} + \frac{\ln(r_o / r_i)}{2\pi k L} + \frac{1}{2\pi r_o h_{cf} L} + \frac{F_i}{2\pi r_i L} + \frac{F_o}{2\pi r_o L}\right]}$$

$$\therefore U_i = \frac{1}{\left[\frac{1}{h_{hf}} + \frac{r_i}{k}\ln\left(\frac{r_o}{r_i}\right) + \frac{r_i}{r_o}\frac{1}{h_{cf}} + F_i + \frac{r_i}{r_o} \times F_o\right]}$$

The heat transfer rate is given by,

$$Q = U_o \times 2\pi r_o L = \frac{1}{\left[\frac{1}{2\pi r_i h_{hf} L} + \frac{\ln(r_o / r_i)}{2\pi k L} + \frac{1}{2\pi r_o h_{cf} L} + \frac{F_i}{2\pi r_i L} + \frac{F_o}{2\pi r_o L}\right]}$$

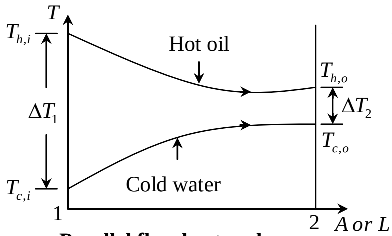
$$\therefore U_o = \frac{1}{\left[\frac{r_o}{r_i}\frac{1}{h_{hf}} + \frac{r_o}{k}\ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_{cf}} + \frac{r_o}{r_i} \times F_i + F_o\right]}$$

In case the tube is thin walled and thermal resistances due to tube wall thickness and fouling are neglected, then overall heat transfer coefficient becomes,

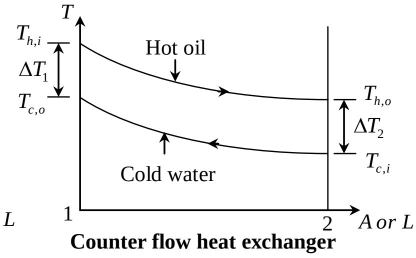
$$U_i = U_o = U = \frac{1}{[1/h_{hf} + 1/h_{cf}]}$$

Question 7.12. A parallel-flow double pipe heat exchanger uses hot oil having specific heat of 1.5 kJ/kg °C to heat water at the rate of 9000 kg/h. The oil enters the heat exchanger at 185 °C and leaves at 135 °C. The inlet and exit temperatures of water are 35 °C and 85 °C, respectively. Take specific heat of water as 4.2 kJ/kg °C. If overall heat transfer coefficient from oil to water is 800 W/m^2 °C, determine the heat transfer area. Also find the increase or decrease in area if the fluids were made to flow in counter direction.

Solution Let $c_h = 1.5\, kJ/kg\,^oC$, $\dot{m}_c = 9000/3600 = 2.5\, kg/s$, $T_{h,i} = 185\,^oC$, $T_{h,o} = 135\,^oC$, $T_{c,i} = 35\,^oC$, $T_{c,o} = 85\,^oC$, $c_c = 4.2\, kJ/kg\,^oC$ and $U = 800 W/m^2\,^oC$.



Parallel flow heat exchanger



Counter flow heat exchanger

(i) For parallel-flow heat exchanger

$$\Delta T_1 = T_{h,i} - T_{c,i} = 185 - 35 = 150\ ^oC \text{ and } \Delta T_2 = T_{h,o} - T_{c,o} = 135 - 85 = 50\ ^oC$$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1 / \Delta T_2)} = \frac{(150-50)}{\ln(150/50)} = 91.024\ ^oC$$

$$Q = UA\Delta T_{lm}$$

$$\dot{m}_c \times c_c \times (T_{c,o} - T_{c,i}) = UA\Delta T_{lm}$$

$$2.5 \times 4.2 \times 10^3 \times (85-35) = 800 \times A \times 91.024$$

$\therefore A = 7.21\, m^2$ **(Ans.)**

(ii) For counter-flow heat exchanger

$$\Delta T_1 = T_{h,i} - T_{c,o} = 185 - 85 = 100\ ^oC \text{ and } \Delta T_2 = T_{h,o} - T_{c,i} = 135 - 35 = 100\ ^oC$$

$$\therefore \Delta T_{lm} = LMTD = \frac{\Delta T_1 + \Delta T_2}{2} = \frac{100+100}{2} = 100\ ^oC$$

Now $Q = UA\Delta T_{lm}$

$$\dot{m}_c \times c_c \times (T_{c,o} - T_{c,i}) = UA\Delta T_{lm}$$

$$2.5 \times 4.2 \times (85-35) = 800 \times A \times 100$$

$$\therefore A = 6.5625\, m^2$$

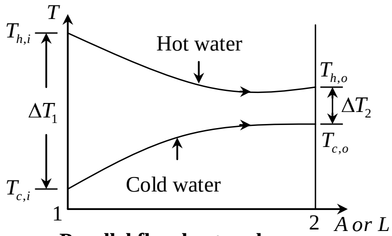
$\therefore \text{Decrease in area} = \frac{7.21 - 6.5625}{7.21} \times 100 = 8.98\%$ **(Ans.)**

Question 7.13. The flow rates of hot and cold water streams flowing through a parallel-flow heat exchanger are 0.25 kg/s and 0.5 kg/s, respectively. The inlet temperatures of the hot and cold water are 80 °C and 25 °C. The exit temperature of hot water is 45 °C. If the heat transfer coefficients on hot and cold sides are 650 W/m^2 °C and 640 W/m^2 °C, respectively, calculate the area of the heat exchanger. Take specific heat of water as 4.186 kJ/kg °C.

Solution Let $\dot{m}_h = 0.25\, kg/s$, $\dot{m}_c = 0.5\, kg/s$, $T_{h,i} = 80\ ^oC$, $T_{c,i} = 25\ ^oC$, $T_{h,o} = 45\ ^oC$, $h_{h,f} = 650 W/m^2\ ^oC$, $h_{c,f} = 640 W/m^2\ ^oC$ and $c_h = c_c = 4.186\, kJ/kg\, ^oC$.

The heat transfer rate is calculated as,

$$Q = \dot{m}_h c_h (T_{h,i} - T_{h,o}) = 0.25 \times 4.186 \times 10^3 \times (80 - 45) = 36627.5\ J/s$$



Parallel flow heat exchanger

First to calculate outlet temperature of cold water as below,

Heat lost by hot water = Heat gained by cold water

$$\dot{m}_h \times c_h \times (T_{h,i} - T_{h,o}) = \dot{m}_c \times c_c \times (T_{c,o} - T_{c,i})$$

$$0.25 \times 4.186 \times 10^3 \times (80 - 45) = 0.5 \times 4.186 \times 10^3 \times (T_{c,o} - 25)$$

$$\therefore T_{c,o} = 42.5\ ^oC$$

For parallel-flow heat exchanger

$\Delta T_1 = T_{h,i} - T_{c,i} = 80 - 25 = 55\ ^oC$ and $\Delta T_2 = T_{h,o} - T_{c,o} = 45 - 42.5 = 2.5\ ^oC$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1 / \Delta T_2)} = \frac{(55 - 2.5)}{\ln(55/2.5)} = 16.98\ ^oC$$

The overall heat transfer coefficient becomes,

$$U = \frac{1}{[1/h_{hf} + 1/h_{cf}]} = \frac{1}{[1/650 + 1/640]} = 322.48\ W/m^2\ ^oC$$

Now $Q = UA\Delta T_{lm}$

$$\therefore A = \frac{Q}{U\Delta T_{lm}} = \frac{36627.5}{322.48 \times 16.98} = 6.69\ m^2$$ **(Ans.)**

Question 7.14. The data for a shell and tube parallel flow heat exchanger to heat the air in the tube by hot exhaust gases flowing in the shell is given as: Thermal conductivity of the tube wall is 350 W/m °C, Heat transferred per second is 43.2 kJ/s, Inside and outside heat transfer coefficients are 125 and 200 W/m^2 °C, respectively, Inlet and outlet temperature of the hot fluid are 455 and 255 °C, respectively, Inlet and outlet temperature of the cold fluid are 65 and 125 °C, respectively, Inside and outside diameters of the tube are 60 and 80 mm, respectively. Evaluate the length of the tube required for the necessary heat transfer to occur.

Solution Refer Figure in the above question.

Let $k = 350\ W/m\ ^oC$, $Q = 43.2\ kJ/s$, $h_{c,f} = 125\ W/m^2\ ^oC$, $h_{h,f} = 200\ W/m^2\ ^oC$, $T_{h,i} = 455\ ^oC$, $T_{h,o} = 255\ ^oC$, $T_{c,i} = 65\ ^oC$, $T_{c,o} = 125\ ^oC$, $d_i = 60\ mm = 0.06\ m$ and $d_o = 80\ mm = 0.08\ m$.

$r_i = d_i/2 = 0.03\,m$ and $r_o = d_o/2 = 0.04\,m$

Let $'L'$ be the length of each tube in meter which is to be determined.

For parallel-flow heat exchanger

$$\Delta T_1 = T_{h,i} - T_{c,i} = 455 - 65 = 395\,^oC \text{ and } \Delta T_2 = T_{h,o} - T_{c,o} = 255 - 125 = 130\,^oC$$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1/\Delta T_2)} = \frac{(395-130)}{\ln(395/130)} = 238.45\,^oC$$

The overall heat transfer coefficient based on outer surface of the inner tube is given by,

Since $$U_o = \frac{1}{\left[\frac{r_o}{r_i}\frac{1}{h_{hf}} + \frac{r_o}{k}\ln\left(\frac{r_o}{r_i}\right) + \frac{1}{h_{cf}}\right]}$$

$$\therefore U_o = \frac{1}{\left[\frac{0.04}{0.03}\times\frac{1}{200} + \frac{0.04}{350}\ln\left(\frac{0.04}{0.03}\right) + \frac{1}{125}\right]} = 68.03\,W/m^{2o}C$$

Now $$Q = UA\Delta T_{lm} = U \times (\pi \times d_o \times L) \times \Delta T_{lm}$$

$$\therefore L = \frac{Q}{U \times \pi \times d_o \times \Delta T_{lm}} = \frac{43.2 \times 10^3}{68.03 \times \pi \times 0.08 \times 238.45} \approx 10.6\,m$$ **(Ans.)**

Question 7.15. The water flows through the inner pipe of a parallel flow heat exchanger which is heated from 20 °C to 70 °C. The hot oil flowing through the annulus is cooled from 200 °C to 100 °C. It is desired to heat the water to a highest possible exit temperature by increasing the length of the heat exchanger. Evaluate the maximum temperature to which the water may be heated.

Answer: Let $T_{c,i} = 20\,^oC$, $T_{c,o} = 70\,^oC$, $T_{h,i} = 200\,^oC$, $T_{h,o} = 100\,^oC$,

Heat lost by hot oil = Heat gained by cold water

$$\dot{m}_h \times c_h \times (T_{h,i} - T_{h,o}) = \dot{m}_c \times c_c \times (T_{c,o} - T_{c,i})$$

$$\dot{m}_h \times c_h \times (200 - 100) = \dot{m}_c \times c_c \times (70 - 20)$$

$$\dot{m}_c \times c_c = 2 \times \dot{m}_h \times c_h$$

Let $'T'$ be the highest possible exit temperature of water which will also be the lowest temperature of oil.

Hence $$\dot{m}_h \times c_h \times (200 - T) = \dot{m}_c \times c_c \times (T - 20)$$

$$\dot{m}_h \times c_h \times (200 - T) = 2 \times \dot{m}_h \times c_h \times (T - 20)$$

$$200 - T = 2 \times (T - 20)$$

$T = 80\ ^oC$ **(Ans.)**

Question 7.16. An oil cooler cools the oil (c = 2.09 kJ/kg °C) flowing at a rate of 1800 kg/h from 80 °C to 40 °C using water (c = 4.2 kJ/kg °C) flowing at a rate of 1800 kg/h at 30 °C. Give your choice for a parallel or counter flow heat exchanger. If the overall heat transfer coefficient is 25 W/m^2 °C, evaluate the surface area of the heat exchanger.

Answer: Let $c_h = 2.09\, kJ/kg\ ^oC$, $\dot{m}_h = 1800\, kg/h = 1800/3600 = 0.5\, kg/s$, $T_{h,i} = 80\ ^oC$, $T_{h,o} = 40\ ^oC$, $c_c = 4.2\, kJ/kg\ ^oC$, $\dot{m}_c = 1800\, kg/h = 1800/3600 = 0.5\, kg/s$ $T_{c,i} = 30\ ^oC$ and $U = 25\, W/m^2\ ^oC$.

Let $'A'$ be the area of heat exchanger which is to be determined.

First to calculate outlet temperature of cold water as below,

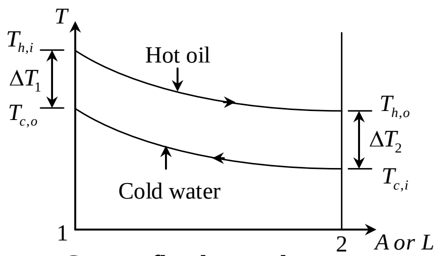
Heat lost by hot oil = Heat gained by cold water

$$\dot{m}_h \times c_h \times (T_{h,i} - T_{h,o}) = \dot{m}_c \times c_c \times (T_{c,o} - T_{c,i})$$

$$0.5 \times 2.09 \times 10^3 \times (80 - 40) = 0.5 \times 4.2 \times 10^3 \times (T_{c,o} - 30)$$

$$\therefore T_{c,o} = 49.9\ ^oC$$

Since outlet temperature of water is higher than the outlet temperature of the oil, therefore, parallel-flow is impossible. Hence, counter-flow cooling arrangement is desired. **(Ans.)**



Counter flow heat exchanger

For counter-flow heat exchanger

$$\Delta T_1 = T_{h,i} - T_{c,o} = 80 - 49.9 = 30.1\ ^oC \text{ and } \Delta T_2 = T_{h,o} - T_{c,i} = 40 - 30 = 10\ ^oC$$

$$\therefore \Delta T_{lm} = LMTD = \frac{(\Delta T_1 - \Delta T_2)}{\ln(\Delta T_1 / \Delta T_2)} = \frac{(30.1 - 10)}{\ln(30.1/10)} = 18.24\ ^oC$$

Now $Q = UA\Delta T_{lm}$

$$\dot{m}_c \times c_c \times (T_{c,o} - T_{c,i}) = UA\Delta T_{lm}$$

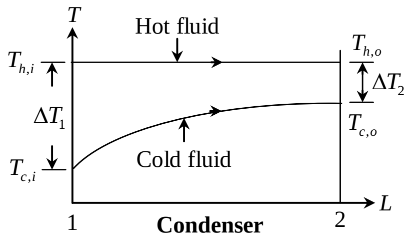
$$0.5 \times 4.2 \times 10^3 \times (49.9 - 30) = 25 \times A \times 18.24$$

$\therefore A = 91.645\, m^2$ **(Ans.)**

Question 7.17. Saturated steam at 120 °C is condensing on the outer surface of a single pass heat exchanger. The overall heat transfer coefficient is 1800 W/m^2 °C. Determine the surface area of heat exchanger capable of heating 0.28 kg/s of water from 20 °C to 90 °C. Also compute the rate of condensation of steam. Take enthalpy of vaporization and specific heat as 2200 kJ/kg and 4.2 kJ/kg °C.

Answer: $T_{h,i} = T_{h,o} = 120\,^oC$, $U = 1800 W/m^2\,^oC$, $\dot{m}_c = 0.28\,kg/s$, $T_{c,i} = 20\,^oC$, $T_{c,o} = 90\,^oC$, $h_{fg} = 2200 kJ/kg$ and $c_c = 4.2\,kJ/kg\,^oC$.

$\Delta T_1 = T_{h,i} - T_{c,i} = 120 - 20 = 100\,^oC$ and $\Delta T_2 = T_{h,o} - T_{c,o} = 120 - 90 = 30\,^oC$



$$\Delta T_{lm} = LMTD = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)} = \frac{100-30}{\ln(100/30)} = 58.14\,^oC$$

Now $\quad Q = \dot{m}_c \times c_c \times (T_{c,o} - T_{c,i}) = 0.28 \times 4.2 \times 10^3 \times (90-20) = 82320\,J/s$

Also $\quad Q = UA\Delta T_{lm}$

$$\therefore A = \frac{Q}{U\Delta T_{lm}} = \frac{82320}{1800 \times 58.14} = 0.787\,m^2$$ **(Ans.)**

Now $\quad Q = \dot{m}_s \times h_{fg}$

$$\therefore \dot{m}_s = \frac{Q}{h_{fg}} = \frac{82320}{2200 \times 10^3} = 0.037\,kg/s$$ **(Ans.)**

Question 7.18. Define (i) Capacity ratio (C), (ii) Effectiveness of heat exchanger (ε) and (iii) Number of transfer units (NTU).

(i) Capacity ratio (C): It is defined as the ratio of the minimum heat capacity to the maximum heat capacity.

$$\therefore C = \frac{C_{\min}}{C_{\max}}$$

In parallel or counter flow heat exchangers, the hot or cold fluid may have the minimum value.

If $\dot{m}_h c_h > \dot{m}_c c_c$ then $C = \dfrac{\dot{m}_c c_c}{\dot{m}_h c_h} = \dfrac{C_c}{C_h}$

If $\dot{m}_h c_h < \dot{m}_c c_c$ then $C = \frac{\dot{m}_h c_h}{\dot{m}_c c_c} = \frac{C_h}{C_c}$

The relative temperature change of the two fluids is inversely related to their heat capacity. That is, fluid with a smaller value of heat capacity experiences the greater change in temperature.

(ii) Effectiveness of heat exchanger (ε): It is defined as the ratio of the actual heat transfer to the maximum possible heat transfer.

$$\varepsilon = \frac{\text{Actual heat transfer}}{\text{Maximum possible heat transfer}} = \frac{Q}{Q_{\max}}$$

The actual heat transfer is given as,

$$Q = \dot{m}_h c_h (T_{h,i} - T_{h,o}) = \dot{m}_c c_c (T_{c,o} - T_{c,i})$$

or $$Q = C_h (T_{h,i} - T_{h,o}) = C_c (T_{c,o} - T_{c,i})$$

Maximum possible heat transfer $(Q_{\max})$ occurs when a fluid of small heat capacity $(C_{\min})$ undergoes maximum temperature change. Both for the parallel and counter flow heat exchangers, the maximum temperature difference available is equal to the inlet temperature of hot fluid $(T_{h,i})$ minus the inlet temperature of cold fluid $(T_{c,i})$, i.e., $\text{Maximum available temeprature difference} = (T_{h,i} - T_{c,i})$.

$$\therefore Q_{\max} = C_{\min}(T_{h,i} - T_{c,i})$$

$$\therefore \varepsilon = \frac{Q}{Q_{\max}} = \frac{C_h (T_{h,i} - T_{h,o})}{C_{\min}(T_{h,i} - T_{c,i})} = \frac{C_c (T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})}$$

Since either the hot or cold fluid may have the minimum value of heat capacity, hence there are two possible below values of effectiveness.

If $C_h < C_c$ then C_h is minimum, $\therefore \varepsilon = \frac{(T_{h,i} - T_{h,o})}{(T_{h,i} - T_{c,i})}$

If $C_c < C_h$ then C_c is minimum, $\therefore \varepsilon = \frac{(T_{c,o} - T_{c,i})}{(T_{h,i} - T_{c,i})}$

Therefore, effectiveness is simply a ratio of the temperature change of the fluid with the smaller heat capacity to the maximum temperature difference available in the heat exchanger.

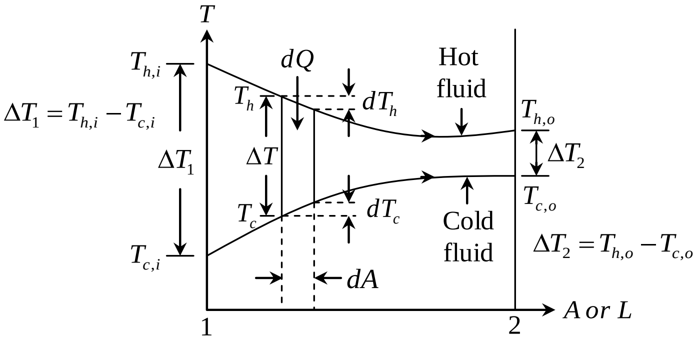
(iii) Number of transfer units (NTU): The number of heat transfer units (NTU) is a dimensionless parameter and is defined below.

$$NTU = \frac{UA}{C_{\min}}$$

It is a measure of the size of heat exchanger. The larger the value of NTU, the closer the heat exchanger reaches its thermodynamic limit (full effectiveness) of operation.

Question 7.19. Derive an expression for effectiveness of a parallel flow heat exchanger.

Answer: A heat exchanger is generally designed by using effectiveness-NTU method when the outlet temperatures of fluids are not given. Figure shows variation of temperatures of the hot and cold fluid streams in a parallel flow double pipe heat exchanger along its length.



Parallel flow heat exchanger

The heat transfer between the cold and hot fluids for an elementary area dA is given by,

$$dQ = UdA(T_h - T_c)$$

The energy balance to differential element between hot and cold fluids gives,

$$dQ = -\dot{m}_h c_h dT_h = -C_h dT_h \Rightarrow dT_h = -\frac{dQ}{C_h}$$

$$dQ = +\dot{m}_c c_c dT_c = +C_c dT_c \Rightarrow dT_c = \frac{dQ}{C_c}$$

$$\therefore d(T_h - T_c) = -dQ\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Substituting $dQ = UdA(T_h - T_c)$ in above equation and rearranging, we get,

$$\frac{d(T_h - T_c)}{(T_h - T_c)} = -UdA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Integrating the above expression from the inlet section 1 to exit section 2, we get,

$$\int_{\Delta T_1}^{\Delta T_2} \frac{d(T_h - T_c)}{(T_h - T_c)} = -U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\int_1^2 dA$$

$$\ln\frac{(T_{h,o} - T_{c,o})}{(T_{h,i} - T_{c,i})} = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right) \qquad \text{(i)}$$

We know that,

$$\varepsilon = \frac{C_h(T_{h,i} - T_{h,o})}{C_{\min}(T_{h,i} - T_{c,i})} = \frac{C_c(T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})}$$

The values of $T_{h,o}$ and $T_{c,o}$ are obtained from above equation as,

$$T_{h,o} = T_{h,i} - \frac{C_{\min}}{C_h}(T_{h,i} - T_{c,i})\varepsilon \text{ and } T_{c,o} = T_{c,i} + \frac{C_{\min}}{C_c}(T_{h,i} - T_{c,i})\varepsilon$$

$$\therefore (T_{h,o} - T_{c,o}) = (T_{h,i} - T_{c,i}) - C_{\min}(T_{h,i} - T_{c,i})\varepsilon\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

or $$\frac{(T_{h,o} - T_{c,o})}{(T_{h,i} - T_{c,i})} = 1 - \varepsilon\, C_{\min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

Now substituting this value in equation (i), we get:

$$\ln\left[1 - \varepsilon\, C_{\min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right] = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

or $$1 - \varepsilon\, C_{\min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right) = \exp\left[-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]$$

or $$\varepsilon = \frac{1 - \exp[-UA(1/C_h + 1/C_c)]}{C_{\min}(1/C_h + 1/C_c)} = \frac{1 - \exp[-UA/C_h(1 + C_h/C_c)]}{C_{\min}/C_h(1 + C_h/C_c)}$$

If C_h is assumed minimum, then $C_h < C_c$ and therefore $C_h = C_{\min}$ and $C_c = C_{\max}$, then

$$\varepsilon = \frac{1 - \exp[-UA/C_{\min}(1 + C_{\min}/C_{\max})]}{(1 + C_{\min}/C_{\max})}$$

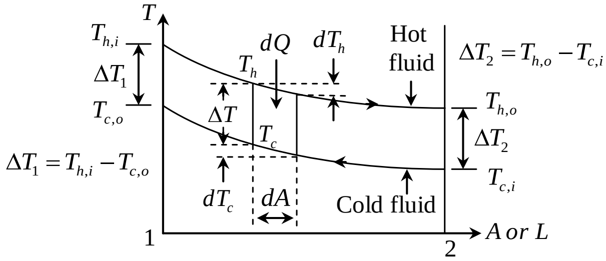
But $NTU = UA/C_{\min}$ and $C = C_{\min}/C_{\max}$

$$\therefore \varepsilon = \frac{1 - \exp[-NTU\,(1 + C)]}{(1 + C)}$$

The same expression will be obtained when the cold fluid has the minimum heat capacity.

Question 7.20. Derive an expression for effectiveness of a counter flow heat exchanger.

Answer: Below Figure shows the variation of temperatures of the hot and cold fluid streams in a counter flow double pipe heat exchanger along its length.



Counter flow heat exchanger

Applying the energy balance to differential element, we get:

$$dQ = -\dot{m}_h c_h dT_h = -C_h dT_h \Rightarrow dT_h = -\frac{dQ}{C_h}$$

$$dQ = -\dot{m}_c c_c dT_c = -C_c dT_c \Rightarrow dT_c = -\frac{dQ}{C_c}$$

$$\therefore d(T_h - T_c) = dQ\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

The heat transfer between the fluids for an elementary area dA is given as $dQ = UdA(T_h - T_c)$. Substituting this value of dQ in above equation and rearranging,

$$\frac{d(T_h - T_c)}{(T_h - T_c)} = UdA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

Integrating the above expression from the inlet section 1 to exit section 2,

$$\int_{\Delta T_1}^{\Delta T_2} \frac{d(T_h - T_c)}{(T_h - T_c)} = U\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\int_1^2 dA$$

$$\ln\frac{T_{h,o} - T_{c,i}}{T_{h,i} - T_{c,o}} = UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

or $$\ln\frac{T_{h,i} - T_{c,o}}{T_{h,o} - T_{c,i}} = -UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

or $$\frac{T_{h,i} - T_{c,o}}{T_{h,o} - T_{c,i}} = \exp\left[-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right] \qquad \text{(i)}$$

We know that,

$$\varepsilon = \frac{C_h(T_{h,i} - T_{h,o})}{C_{\min}(T_{h,i} - T_{c,i})} = \frac{C_c(T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})}$$

The values of $T_{h,o}$ and $T_{c,o}$ are obtained from above equation as,

$$T_{h,o} = T_{h,i} - \frac{C_{\min}}{C_h}(T_{h,i} - T_{c,i})\varepsilon \text{ and } T_{c,o} = T_{c,i} + \frac{C_{\min}}{C_c}(T_{h,i} - T_{c,i})\varepsilon$$

Substituting these values in Equation (i), we get:

$$\frac{T_{h,i} - T_{c,i} - (C_{\min}/C_c)(T_{h,i} - T_{c,i})\varepsilon}{T_{h,i} - (C_{\min}/C_h)(T_{h,i} - T_{c,i})\varepsilon - T_{c,i}} = \exp[-UA(1/C_c - 1/C_h)]$$

or $$\frac{(T_{h,i} - T_{c,i})[1 - (C_{\min}/C_c)\varepsilon]}{(T_{h,i} - T_{c,i})[1 - (C_{\min}/C_h)\varepsilon} = \exp[-UA(1/C_c - 1/C_h)]$$

or $$\frac{[1-(C_{\min}/C_c)\varepsilon]}{[1-(C_{\min}/C_h)\varepsilon} = \exp[-UA(1/C_c - 1/C_h)]$$

$$1-\frac{C_{\min}}{C_c}\varepsilon = \exp[-UA(1/C_c - 1/C_h)] - (C_{\min}/C_h)\varepsilon.\exp[-UA(1/C_c - 1/C_h)]$$

$$1-\exp[-UA(1/C_c - 1/C_h)] = \frac{C_{\min}}{C_c}\varepsilon - \frac{C_{\min}}{C_h}\varepsilon.\exp[-UA(1/C_c - 1/C_h)]$$

$$1-\exp[-UA(1/C_c - 1/C_h)] = \varepsilon[(C_{\min}/C_c) - (C_{\min}/C_h)\exp\{-UA(1/C_c - 1/C_h)\}]$$

$$\varepsilon = \frac{1-\exp[-UA(1/C_c - 1/C_h)]}{C_{\min}[1/C_c - 1/C_h \exp\{-UA/C_c(1-C_c/C_h)\}]}$$

$$\varepsilon = \frac{1-\exp[-UA/C_c\left(1-C_c/C_h\right)}{C_{\min}/C_c[1-(C_c/C_h)\exp\{-UA/C_c(1-C_c/C_h)\}}$$

If C_c is assumed minimum, then $C_c < C_h$ and therefore $C_c = C_{\min}$ and $C_h = C_{\max}$, then

$$\varepsilon = \frac{1-\exp[-UA/C_{\min}(1-C_{\min}/C_{\max})]}{C_{\min}/C_{\min}[1-(C_{\min}/C_{\max})\exp\{-UA/C_{\min}(1-C_{\min}/C_{\max})\}]}$$

But $NTU = UA/C_{\min}$ and $C = C_{\min}/C_{\max}$

$$\therefore \varepsilon = \frac{1-\exp[-NTU\left(1-C\right)]}{1-C\exp[-NTU(1-C)]}$$

The same expression will be obtained when the hot fluid has the minimum heat capacity.

Question 7.21. Derive expression for the effectiveness of a condenser.

Answer: The rate of heat transfer in a condenser is given by,

$$Q = C_h(T_{h,i} - T_{h,o})$$

$$\therefore C_h = \frac{Q}{T_{h,i} - T_{h,o}} = \frac{Q}{0} = \infty = C_{\max} \qquad (\because T_{h,i} = T_{h,o})$$

$$\therefore C = \frac{C_{\min}}{C_{\max}} = \frac{C_{\min}}{\infty} = 0$$

Now $$\varepsilon = \frac{1-\exp[-NTU(1+C)]}{(1+C)}$$ (Parallel-flow)

Substituting $C = 0$ in the above expression, we get:

$$\varepsilon = 1 - \exp(-NTU)$$

Now $$\varepsilon = \frac{1-\exp[-NTU\left(1-C\right)]}{1-C\exp[-NTU(1-C)]}$$ (Counter-flow)

Substituting $C = 0$ in the above expression, we get:

$$\varepsilon = 1 - \exp(-NTU)$$

Hence for both types of flow, the effectiveness of a condenser is given by,

$$\varepsilon = 1 - \exp(-NTU)$$

Question 7.22. Derive expression for the effectiveness of an evaporator.

Answer: The rate of heat transfer in an evaporator is given by,

$$Q = C_c (T_{c,o} - T_{c,i})$$

$$\therefore C_c = \frac{Q}{T_{c,o} - T_{c,i}} = \frac{Q}{0} = \infty = C_{\max} \qquad (\because T_{c,o} = T_{c,i})$$

$$\therefore C = \frac{C_{\min}}{C_{\max}} = \frac{C_{\min}}{\infty} = 0$$

Now substituting $C = 0$ in the expressions for effectiveness of an evaporator for parallel and counter flow conditions, we get:

$$\varepsilon = \frac{1 - \exp[-NTU(1+0)]}{(1+0)} = 1 - \exp(-NTU) \qquad \text{(Parallel-flow)}$$

$$\varepsilon = \frac{1 - \exp[-NTU(1-0)]}{1 - 0\exp[-NTU(1-0)]} = 1 - \exp(-NTU) \qquad \text{(Counter-flow)}$$

Hence for both types of flow, the effectiveness of an evaporator is given by,

$$\varepsilon = 1 - \exp(-NTU)$$

Question 7.23. Derive expressions for the effectiveness of a regenerator for parallel and counter flow conditions.

Answer: In the case of typical regenerators, $C_{\min} = C_{\max}$

$$\therefore C = \frac{C_{\min}}{C_{\max}} = 1$$

(i) $\varepsilon = \dfrac{1 - \exp[-NTU(1+C)]}{(1+C)}$ (Parallel-flow)

Substituting $C = 1$ in the above expression, we get:

$$\varepsilon = \frac{1 - \exp[-NTU(1+1)]}{(1+1)} = \frac{1 - \exp(-2NTU)}{2}$$

(ii) $\varepsilon = \dfrac{1 - \exp[-NTU(1-C)]}{1 - C\exp[-NTU(1-C)]}$ (Counter-flow)

Substituting $C = 1$ in the above expression, we get:

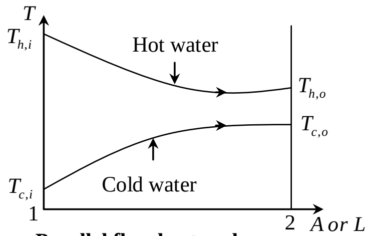
$$\varepsilon = \frac{1-\exp[-NTU(1-1)]}{1-1\times\exp[-NTU(1-1)]} = \frac{0}{0}$$, which is indeterminate.

Therefore, applying L, Hospital's rule, we get:

$$\varepsilon = \underset{C\to 1}{Limit}\frac{\frac{\partial}{\partial C}[1-\exp\{-NTU(1-C)\}]}{\frac{\partial}{\partial C}[1-C\exp\{-NTU(1-C)\}]} = \frac{NTU}{1+NTU}$$

Question 7.24. In a double pipe parallel flow heat exchanger hot water flows at a rate of 50040 kg/h and gets cooled from 95 °C to 65 °C. The cooling water enters at 30 °C in the exchanger at a rate of 50040 kg/h. Determine the heat transfer area and the effectiveness if overall heat transfer coefficient is 2270 W/m^2 °C and specific heat of hot and cold water is 4.2 kJ/kg °C.

Solution Let $\dot{m}_h = 50040/3600 = 13.9\ kg/s$, $T_{h,i} = 95\ ^oC$, $T_{h,o} = 65\ ^oC$, $T_{c,i} = 30\ ^oC$, $\dot{m}_c = 50040/3600 = 13.9\ kg/s$, $U = 2270\ W/m^2\ ^oC$, and $c_h = c_c = 4.2\ kJ/kg\ ^oC$.



Parallel flow heat exchanger

First to calculate outlet temperature of cold water as below,

Heat lost by hot water = Heat gained by cold water

$$\dot{m}_h c_h (T_{h,i} - T_{h,o}) = \dot{m}_c c_c (T_{c,o} - T_{c,i})$$

$$13.9\times 4.2\times 10^3\times(95-65) = 13.9\times 4.2\times 10^3\times(T_{c,o}-30)$$

$$\therefore T_{c,o} = 60\ ^oC$$

For parallel-flow heat exchanger

$\Delta T_1 = T_{h,i} - T_{c,i} = 95-30 = 65\ ^oC$ and $\Delta T_2 = T_{h,o} - T_{c,o} = 65-60 = 5\ ^oC$

$$\therefore \Delta T_{lm} = LMTD = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)} = \frac{65-5}{\ln(65/5)} = 23.39\ ^oC$$

Now $Q = UA\Delta T_{lm}$

Thus $A = \frac{Q}{U\Delta T_{lm}} = \frac{\dot{m}_h c_h (T_{h,i} - T_{h,o})}{U\Delta T_{lm}}$

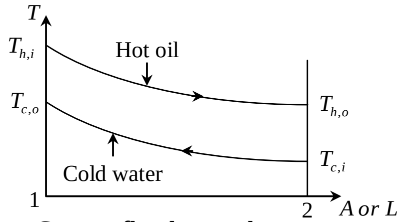
$\therefore A = \frac{13.9\times 4.2\times 10^3\times(95-65)}{2270\times 23.39} = 32.986\ m^2$ **(Ans.)**

Now $\varepsilon = \dfrac{Q}{Q_{\max}} = \dfrac{C_h(T_{h,i} - T_{h,o})}{C_{\min}(T_{h,i} - T_{c,i})} = \dfrac{(T_{h,i} - T_{h,o})}{(T_{h,i} - T_{c,i})}$ $(\because C_h = C_{\min} = C_c)$

$$\therefore \varepsilon = \frac{95-65}{95-30} = 0.4615$$ **(Ans.)**

Question 7.25. A counter flow heat exchanger is used to cool 0.56 kg/s of oil (c = 2.5 kJ/kg °C) from 105 °C to 30 °C. The cold water enters at 15 °C and comes out at 80 °C. Use NTU-effectiveness method for determining the water flow rate, effectiveness and heat transfer area of heat exchanger if the overall heat transfer coefficient is 1500 W/m^2 °C and specific heat for water is 4.2 kJ/kg °C.

Solution Let $\dot{m}_h = 0.56\, kg/s$, $c_h = 2.5\, kJ/kg\,^oC$ $T_{h,i} = 105\,^oC$, $T_{h,o} = 30\,^oC$, $T_{c,i} = 15\,^oC$, $T_{c,o} = 80\,^oC$, $U = 1500\, W/m^2\,^oC$ and $c_c = 4.2\, kJ/kg\,^oC$.



Counter flow heat exchanger

(i) $$\dot{m}_h c_h (T_{h,i} - T_{h,o}) = \dot{m}_c c_c (T_{c,o} - T_{c,i})$$

$$0.56 \times 2.5 \times 10^3 \times (105 - 30) = \dot{m}_c \times 4.2 \times 10^3 \times (80 - 15)$$

$$\therefore \dot{m}_c = 0.385\, kg/s$$ **(Ans.)**

(ii) $$C_c = \dot{m}_c c_c = 0.385 \times 4.2 \times 10^3 = 1617\, J/s\,^oC \quad (C_{\max})$$

$$C_h = \dot{m}_h c_h = 0.56 \times 2.5 \times 10^3 = 1400\, J/s\,^oC \quad (C_{\min})$$

$$C = \frac{C_{\min}}{C_{\max}} = \frac{1400}{1617} = 0.866$$

$$\varepsilon = \frac{T_{h,i} - T_{h,o}}{T_{h,i} - T_{c,i}} = \frac{105-30}{105-15} = 0.833$$ **(Ans.)**

(iii) $$\varepsilon = \frac{1 - \exp[-NTU(1-C)]}{1 - C\exp[-NTU(1-C)]}$$ (Counter-flow)

$$\varepsilon - \varepsilon C \exp[-NTU(1-C)] = 1 - \exp[-NTU(1-C)]$$

$$\varepsilon - 1 = \varepsilon C \exp[-NTU(1-C)] - \exp[-NTU(1-C)]$$

$$\varepsilon - 1 = \exp[-NTU(1-C)] \times (\varepsilon C - 1)$$

$$\exp[-NTU(1-C)] = \frac{\varepsilon - 1}{\varepsilon C - 1}$$

$$\exp[-NTU(1-0.866)] = \frac{0.833-1}{0.833\times 0.866-1}$$

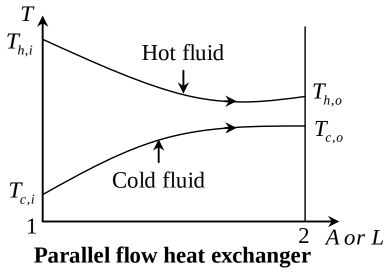
$$-0.134NTU = \ln\left(\frac{0.167}{0.2786}\right)$$

$$\therefore NTU = 3.82$$

$$NTU = \frac{UA}{C_{\min}} \Rightarrow A = \frac{NTU\times C_{\min}}{U} = \frac{3.82\times 1400}{1500} = 3.565\, m^2$$ **(Ans.)**

Question 7.26. A hot fluid at 200 °C enters a parallel flow heat exchanger at a mass flow rate of 10008 kg/h. Its specific heat is 2000 J/kg °C. It is cooled by another fluid entering at 25 °C with a mass flow rate of 2502 kg/h having specific heat of 400 J/kg °C. Using NTU-effectiveness method determine the outlet temperature of the hot fluid if the overall heat transfer coefficient based on outside area of 20.2 m^2 is 255 W/m °C.

Solution Let $T_{h,i} = 200\,^oC$, $\dot{m}_h = 10008\, kg/h = 2.78\, kg/s$, $c_h = 2000\, J/kg\,^oC$, $T_{c,i} = 25\,^oC$, $\dot{m}_c = 2502\, kg/h = 0.695\, kg/s$, $c_c = 400\, J/kg\,^oC$, $A = 20.2\, m^2$ and $U = 255\, W/m^2\,^oC$.



Parallel flow heat exchanger

$$C_h = \dot{m}_h c_h = 2.78\times 2000 = 5560\, J/s\,^oC \quad (C_{\max})$$

$$C_c = \dot{m}_c c_c = 0.695\times 400 = 278\, J/s\,^oC \quad (C_{\min})$$

$$C = \frac{C_{\min}}{C_{\max}} = \frac{278}{5560} = 0.05$$

$$NTU = \frac{UA}{C_{\min}} = \frac{255\times 20.2}{278} = 18.53$$

$$\varepsilon = \frac{1-\exp[-NTU(1+C)]}{(1+C)} = \frac{1-\exp[-18.53(1+0.05)]}{(1+0.05)} = 0.952$$

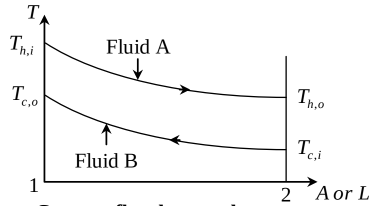
Also $$\varepsilon = \frac{Q}{Q_{\max}} = \frac{C_h(T_{h,i}-T_{h,o})}{C_{\min}(T_{h,i}-T_{c,i})}$$

Thus $$0.952 = \frac{5560\times(200-T_{h,o})}{278\times(200-25)}$$

$$\therefore T_{h,o} = 200 - \frac{0.952 \times 278 \times (200-25)}{5560} = 191.67\ ^oC$$ **(Ans.)**

Question 7.27. Two fluids A and B exchange heat in a counter flow heat exchanger. Fluid 'A' (specific heat 1000 J/kg °C) enters at 420 °C having a mass flow rate of 1 kg/s. Fluid 'B' (specific heat 4000 J/kg °C) enters at 20 °C which also has a mass flow rate of 1 kg/s. The effectiveness of heat exchanger is 0.75. Determine the exit temperature of fluid and heat transfer rate.

Solution Let $c_h = 1000\, J/kg\,^oC$ $T_{h,i} = 420\,^oC$, $\dot{m}_h = 1\, kg/s$, $c_c = 4000\, J/kg\,^oC$, $T_{c,i} = 20\,^oC$, $\dot{m}_c = 1\, kg/s$ and $\varepsilon = 0.75$.



Counter flow heat exchanger

Since only inlet temperatures are given for both fluids, therefore, we have to use NTU-effectiveness method.

$$C_h = \dot{m}_h c_h = 1 \times 1000 = 1000\, J/s\,^oC \quad (C_{\min})$$

$$C_c = \dot{m}_c c_c = 1 \times 4000 = 4000\, J/s\,^oC \quad (C_{\max})$$

$$\varepsilon = \frac{Q}{Q_{\max}} = \frac{C_c(T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})} = \frac{4000 \times (T_{c,o} - 20)}{1000 \times (420 - 20)}$$

$$0.75 = \frac{4000 \times (T_{c,o} - 20)}{1000 \times (420 - 20)} \Rightarrow \therefore T_{c,o} = 95\,^oC$$ **(Ans.)**

$$Q = C_c(T_{c,o} - T_{c,i}) = \frac{4000 \times (95 - 20)}{1000} = 300\, kW$$ **(Ans.)**

Question 7.28. Two fluids A and B exchange heat in a counter flow heat exchanger. Fluid 'A' (specific heat 3.5 kJ/kg °C) enters at 645 °C having a mass flow rate of 16 kg/s. Fluid 'B' (specific heat 4.2 kJ/kg °C) enters at 100 °C which has a mass flow rate of 20 kg/s. Using NTU-effectiveness method determine the effectiveness and outlet temperature of the hot fluid if the overall heat transfer coefficient based on outside area of 45 m^2 is 950 W/m^2 °C.

Solution Refer Figure given in Question 27. Let $c_h = 3.5\, kJ/kg\,^oC$, $T_{h,i} = 645\,^oC$, $\dot{m}_h = 16\, kg/s$, $c_c = 4.2\, kJ/kg\,^oC$, $T_{c,i} = 100\,^oC$, $\dot{m}_c = 20\, kg/s$, $A = 45\, m^2$ and $U = 950\, W/m^2\ ^oC$.

$$C_c = \dot{m}_c c_c = 20 \times 4.2 = 84\, kW/^oC \quad (C_{\max})$$

$$C_h = \dot{m}_h c_h = 16 \times 3.5 = 56\, kW/^oC \quad (C_{\min})$$

$$C = \frac{C_{\min}}{C_{\max}} = \frac{56}{84} = 0.67$$

$$NTU = \frac{UA}{C_{\min}} = \frac{950 \times 45}{56 \times 10^3} = 0.76$$

$$\varepsilon = \frac{1 - \exp[-NTU(1-C)]}{1 - C\exp[-NTU(1-C)]}$$ (Counter-flow)

$$\therefore \varepsilon = \frac{1 - \exp[-0.76(1-0.67)]}{1 - 0.67 \times \exp[-0.76(1-0.67)]} = 0.4635$$ **(Ans.)**

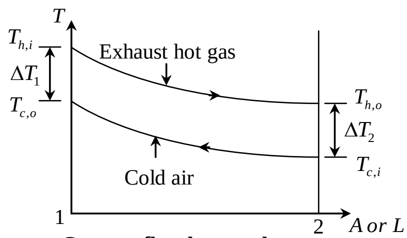
Also $\varepsilon = \frac{T_{h,i} - T_{h,o}}{T_{h,i} - T_{c,i}}$ $(\because C_h = C_{\min})$

$$T_{h,o} = T_{h,i} - \varepsilon \times (T_{h,i} - T_{c,i})$$

$$\therefore T_{h,o} = 645 - 0.4635 \times (645 - 100) = 392.4\ ^oC$$ **(Ans.)**

Question 7.29. A counter flow heat exchanger is to heat air entering at 673 K with a flow rate of 6.25 kg/s by the exhaust gas entering at 1073 K with a flow rate of 4.25 kg/s. The specific heat for both air and exhaust gas is given as 1.2 kJ/kg K. If the outlet temperature of the air is 823 K and overall heat transfer coefficient is 100 W/m^2 K, determine the outlet temperature of the gas, heat transfer surface area and the NTU.

Solution Let $T_{c,i} = 673K$, $\dot{m}_c = 6.25\,kg/s$, $T_{h,i} = 1073K$, $\dot{m}_h = 4.25\,kg/s$, $c_c = c_h = 1.2\,kJ/kg\,K$, $T_{c,o} = 823K$ and $U = 100\,W/m^2\,K$.



Counter flow heat exchanger

$$C_c = \dot{m}_c c_c = 6.25 \times 1.2 = 7.5\,kW/K \quad (C_{\max})$$

$$C_h = \dot{m}_h c_h = 4.25 \times 1.2 = 5.4\,kW/K \quad (C_{\min})$$

(i) First to calculate outlet temperature of exhaust gas as below,

Heat lost by hot exhaust gas = Heat gained by cold air

$$\dot{m}_h \times c_h \times (T_{h,i} - T_{h,o}) = \dot{m}_c \times c_c \times (T_{c,o} - T_{c,i})$$

$$4.25 \times 1.2 \times (1073 - T_{h,o}) = 6.25 \times 1.2 \times (823 - 673)$$

$$\therefore T_{h,o} = 889.18\,K$$ **(Ans.)**

(ii) $\Delta T_1 = T_{h,i} - T_{c,o} = 1073 - 823 = 250\,K$

$$\Delta T_2 = T_{h,o} - T_{c,i} = 889.18 - 673 = 216.18\,K$$

$$\therefore \Delta T_{lm} = LMTD = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1 / \Delta T_2)} = \frac{250 - 216.18}{\ln(250/216.18)} = 232.68\,K$$

Now $Q = UA\Delta T_{lm}$

$$\dot{m}_c \times c_c \times (T_{c,o} - T_{c,i}) = UA\Delta T_{lm}$$

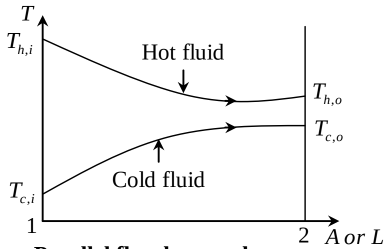
$$6.25 \times 1.2 \times 10^3 \times (823 - 673) = 100 \times A \times 232.68$$

$\therefore A = 48.35\,m^2$ **(Ans.)**

(iii) $NTU = \dfrac{UA}{C_{\min}} = \dfrac{100 \times 48.35}{5.4 \times 10^3} = 0.895$ **(Ans.)**

Question 7.30. A hot fluid at 398 K enters a parallel flow heat exchanger at a mass flow rate of 5.6 kg/s. Its specific heat is 3.4 kJ/kg K. It is cooled by a cold fluid entering at 298 K with a mass flow rate of 13.9 kg/s having specific heat of 4.2 kJ/kg K. Using NTU-effectiveness method determine the outlet temperature of the hot and cold fluids if the overall heat transfer coefficient based on outside area of 10.5 m^2 is 1055 W/m K.

Solution Let $T_{h,i} = 398\,K$, $\dot{m}_h = 5.6\,kg/s$, $c_h = 3.4\,kJ/kg\,K$, $T_{c,i} = 298\,K$, $\dot{m}_c = 13.9\,kg/s$, $c_c = 4.2\,kJ/kg\,K$, $A = 10.5\,m^2$ and $U = 1055\,W/m^2\,K$.



Parallel flow heat exchanger

$$C_h = \dot{m}_h c_h = 5.6 \times 3.4 = 19.04\,kJ/K \quad (C_{\min})$$

$$C_c = \dot{m}_c c_c = 13.9 \times 4.2 = 58.38\,kJ/K \quad (C_{\max})$$

$$C = \frac{C_{\min}}{C_{\max}} = \frac{19.04}{58.38} = 0.326$$

$$NTU = \frac{UA}{C_{\min}} = \frac{1055 \times 10.5}{19.04 \times 10^3} = 0.582$$

$$\varepsilon = \frac{1 - \exp[-NTU\,(1 + C)]}{(1 + C)}$$

$\therefore \varepsilon = \dfrac{1 - \exp[-0.582(1 + 0.326)]}{(1 + 0.326)} = 0.405$ **(Ans.)**

Also $\varepsilon = \dfrac{T_{h,i} - T_{h,o}}{T_{h,i} - T_{c,i}}$ $[\because C_h = C_{\min}]$

Thus $0.405 = \dfrac{398 - T_{h,o}}{398 - 298}$

$\therefore T_{h,o} = 357.5\, K$ **(Ans.)**

$Q = \dot{m}_h c_h (T_{h,i} - T_{h,o}) = 5.6 \times 3.4 \times (398 - 357.5) = 771.12\, kW$

Also $Q = \dot{m}_c c_c (T_{c,o} - T_{c,i})$

$771.12 = 13.9 \times 4.2 \times (T_{c,o} - 298)$

$\therefore T_{c,o} = 311.21\, K$ **(Ans.)**